Nankai Tracts in Mathematics Vol. 4

LECTURES ON CHERN-WEIL THEORY AND WITTEN DEFORMATIONS

Weiping Zhang

World Scientific

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LECTURES ON CHERN-WEIL THEORY AND WITTEN DEFORMATIONS

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Dedicated to my teachers Jean-Michel Bismut and Shiing-Shen Chern

Preface

These lecture notes are based on the notes of a graduate course of differential geometry I taught at the Nankai Institute of Mathematics. It consists of two parts: the first geometric part contains an introduction to the geometric theory of characteristic classes due to Shiing-shen Chern and André Weil, as well as a proof of the Gauss-Bonnet-Chern theorem based on the Mathai-Quillen construction of Thom forms; while the second part, which is analytic in nature, contains analytic proofs of the Poincaré-Hopf index formula as well as the Morse inequalities based on deformations introduced by Edward Witten.

We hope this book can serve as a text book to cover materials not generally contained in an introductory course in differential geometry. With this reason, we have not tried hard to make this book being completely self-contained. However, we will give detailed references when (possibly) nonstandard results will be quoted. On the other hand, we have tried to make each chapter in the text to be relatively independent from the other chapters. As a result, we will list the references of each chapter at the end of that chapter.

We will work in smooth (i.e. C^∞) category throughout this book.

I would like to thank Dr. Huitao Feng for taking the preliminary notes for my lectures. Part of these notes were prepared during a short visit to the Institute of Mathematics of Fudan University in May, 2000, and during my visit to the Department of Mathematics of MIT for the Spring semester of 2001. I would like to thank Professor Jiaxing Hong of Fudan University for kind hospitality. I am also grateful to Professors Richard Melrose and Gang Tian for arranging my visit to MIT, and to MIT for financial support.

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Chapter 1

Chern-Weil Theory for Characteristic Classes

The theory of characteristic classes of vector bundles over smooth manifolds plays important roles in topology and geometry. The book of Milnor and Stasheff [MS] contains a beautiful introduction to the topological aspects of this theory.

This chapter contains an introduction to the geometric aspects of this theory, which was developed by Shiing-shen Chern and André Weil.

1.1 Review of the de Rham Cohomology Theory

This section contains a brief review of the de Rham cohomology theory. For more details, we recommend the standard book of Bott and Tu [BoT].

Let M be a smooth closed manifold. Let TM (resp. T^*M) denote the tangent (resp. cotangent) vector bundle of M. We denote by $\Lambda^*(T^*M)$ the (complex) exterior algebra bundle of T^*M, and

$$\Omega^*(M) := \Gamma\left(\Lambda^*\left(T^*M\right)\right)$$

the space of smooth sections of $\Lambda^*(T^*M)$. In particular, for any integer p such that $0 \leq p \leq \dim M$, we denote by

$$\Omega^p(M) := \Gamma\left(\Lambda^p\left(T^*M\right)\right)$$

the space of smooth p-forms over M.

Let

$$d : \Omega^*(M) \longrightarrow \Omega^*(M)$$

denote the exterior differential operator. Then d maps a p-form to a $(p+1)$-form. Furthermore, there holds the following important formula,

$$d^2 = 0. \tag{1.1}$$

We adopt the convention that both $\Omega^{-1}(M)$ and $\Omega^{\dim M+1}(M)$ are spaces of zero.

From (1.1), one finds that for any integer p such that $0 \leq p \leq \dim M$, one has

$$d\,\Omega^p(M) \subset \ker d|_{\Omega^{p+1}(M)},$$

which leads to the definition of the de Rham complex as well as its associated cohomology: de Rham cohomology.

Definition 1.1 The **de Rham complex** $(\Omega^*(M), d)$ is the complex defined by

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim M}(M) \longrightarrow 0.$$

Definition 1.2 For any integer p such that $0 \leq p \leq \dim M$, the p-th **de Rham cohomology** of M (with complex coefficient) is defined by

$$H^p_{\mathrm{dR}}(M; \mathbf{C}) = \frac{\ker d|_{\Omega^p(M)}}{d\Omega^{p-1}(M)}.$$

The (total) **de Rham cohomology** of M is then defined as

$$H^*_{\mathrm{dR}}(M; \mathbf{C}) = \bigoplus_{p=0}^{\dim M} H^p_{\mathrm{dR}}(M; \mathbf{C}).$$

From the definition of the de Rham cohomology, one sees that any *closed* differential form ω on M, that is, any element $\omega \in \Omega^*(M)$ such that $d\omega = 0$, determines a cohomology class $[\omega] \in H^*_{\mathrm{dR}}(M, \mathbf{C})$. Moreover, two closed differential forms ω, ω' on M determine the same cohomology class if and only if there exists a differential form η such that $\omega - \omega' = d\eta$.

If ω, ω' are two closed differential forms on M and a is a constant function on M, then one verifies easily that the following identities in $H^*_{\mathrm{dR}}(M; \mathbf{C})$ hold,

$$[a\omega] = a[\omega], \qquad [\omega + \omega'] = [\omega] + [\omega'].$$

Moreover, for any two differential forms η, η' on M, one verifies that

$$(\omega + d\eta) \wedge (\omega' + d\eta') = \omega \wedge \omega' + d\left(\eta \wedge \omega' + \eta \wedge d\eta' + (-1)^{\deg \omega}\omega \wedge \eta'\right).$$

Thus the cohomology class $[\omega \wedge \omega']$ depends only on $[\omega]$ and $[\omega']$. We denote it by $[\omega] \cdot [\omega']$ and call it the product of $[\omega]$ and $[\omega']$.

If ω'' is a third closed differential form on M, then one can verify that

$$([\omega] + [\omega']) \cdot [\omega''] = [\omega] \cdot [\omega''] + [\omega'] \cdot [\omega''].$$

From the above discussion, one sees that the de Rham cohomology of M carries a natural ring structure.

The importance of the de Rham cohomology lies in the **de Rham theorem** which we state as follows, and which we refer to the book [BoT] for a proof.

Theorem 1.3 *If M is a smooth closed orientable manifold, then for any integer p such that $0 \leq p \leq \dim M$,*

(i) $\dim H^p_{\mathrm{dR}}(M; \mathbf{C}) < +\infty$;

(ii) $H^p_{\mathrm{dR}}(M; \mathbf{C})$ *is canonically isomorphic to* $H^p_{\mathrm{Sing}}(M; \mathbf{C})$, *the p-th singular cohomology of M.*

1.2 Connections on Vector Bundles

We still refer to the book [BoT] for the basic theory of vector bundles over smooth manifolds.

Let $E \to M$ be a smooth complex vector bundle over a smooth compact manifold M. We denote by $\Omega^*(M; E)$ the space of smooth sections of the tensor product vector bundle $\Lambda^*(T^*M) \otimes E$ obtained from $\Lambda^*(T^*M)$ and E,

$$\Omega^*(M; E) := \Gamma(\Lambda^*(T^*M) \otimes E).$$

A connection on E may be thought of, in some sense, as an extension of the exterior differential operator d to include the coefficient E.

Definition 1.4 A **connection** ∇^E on E is a $\mathbf{C}$-linear operator $\nabla^E : \Gamma(E) \to \Omega^1(M; E)$ such that for any $f \in C^\infty(M)$, $X \in \Gamma(E)$, the following

Leibniz rule holds,

$$\nabla^E(fX) = (df)X + f\nabla^E X.$$

The existence of a connection on a vector bundle can be proved easily by using the method of partitions of unity. Certainly, there are a lot of connections on a vector bundle if one does not impose further conditions (In fact they form an infinite dimensional affine space). In many cases it is important in geometry to find and study connections verifying various specific geometric conditions.

If $X \in \Gamma(TM)$ is a smooth section of TM, then a connection ∇^E induces canonically a map

$$\nabla_X^E : \Gamma(E) \longrightarrow \Gamma(E)$$

via the contraction between TM and T^*M. We call it the **covariant derivative** of ∇^E along X.

Just like the exterior differential operator d, a connection ∇^E can be extended canonically to a map, which we still denote by ∇^E,

$$\nabla^E : \Omega^*(M; E) \longrightarrow \Omega^{*+1}(M; E)$$

such that for any $\omega \in \Omega^*(M)$, $X \in \Gamma(E)$,

$$\nabla^E : \omega X \mapsto (d\omega)X + (-1)^{\deg \omega}\omega \wedge \nabla^E X. \tag{1.2}$$

1.3 The Curvature of a Connection

The importance of the concept of a connection lies in its curvature.

Definition 1.5 The **curvature** R^E of a connection ∇^E is defined by

$$R^E = \nabla^E \circ \nabla^E : \Gamma(E) \to \Omega^2(M; E),$$

which, for brevity, we will write $R^E = (\nabla^E)^2$.

The following property of curvature is of critical importance.

Proposition 1.6 *The curvature R^E is $C^\infty(M)$-linear. That is, for any $f \in C^\infty(M)$ and $X \in \Gamma(E)$, one has*

$$R^E(fX) = f\, R^E X.$$

Proof. By using (1.1) and (1.2), one deduces that

$$R^E(fX) = \nabla^E\left((df)X + f\nabla^E X\right)$$

$$= (-1)^{\deg df} df \wedge \nabla^E X + df \wedge \nabla^E X + f\left(\nabla^E\right)^2 X$$

$$= f\, R^E X.$$ □

Let $\mathrm{End}(E)$ denote the vector bundle over M formed by the fiberwise endomorphisms of E.

From Proposition 1.6, one sees that R^E may be thought of as an element of $\Gamma(\mathrm{End}(E))$ with coefficients in $\Omega^2(M)$. In other words,

$$R^E \in \Omega^2(M; \mathrm{End}(E)).$$

To give a more precise formula, if X, $Y \in \Gamma(TM)$ are two smooth sections of TM, then $R^E(X,Y)$ is an element in $\Gamma(\mathrm{End}(E))$ given by

$$R^E(X,Y) = \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]}, \tag{1.3}$$

where $[X,Y] \in \Gamma(TM)$ is the **Lie bracket** of X and Y defined by the formula that for any $f \in C^\infty(M)$,

$$[X,Y]f = X(Yf) - Y(Xf) \in C^\infty(M).$$

Finally, in view of the composition of the endomorphisms, one sees that for any integer $k \geq 0$,

$$\left(R^E\right)^k = \overbrace{R^E \circ \cdots \circ R^E}^{k} : \Gamma(E) \longrightarrow \Omega^{2k}(M; E)$$

is a well-defined element lying in $\Omega^{2k}(M; \mathrm{End}(E))$.

1.4 Chern-Weil Theorem

We continue the discussion in the above section.

For any smooth section A of the bundle of endomorphisms, $\mathrm{End}(E)$, the fiberwise trace of A forms a smooth function on M. We denote this function by $\mathrm{tr}[A]$. This further induces the map

$$\mathrm{tr} : \Omega^*(M; \mathrm{End}(E)) \longrightarrow \Omega^*(M)$$

such that for any $\omega \in \Omega^*(M)$ and $A \in \Gamma(\mathrm{End}(E))$,

$$\mathrm{tr} : \omega A \mapsto \omega \, \mathrm{tr}[A].$$

We still call it the function of trace.

We also extend the Lie bracket operation on $\mathrm{End}(E)$ to $\Omega^*(M; \mathrm{End}(E))$ as follows: if ω, $\eta \in \Omega^*(M)$ and A, $B \in \Gamma(\mathrm{End}(E))$, then we use the convention that

$$[\omega A, \eta B] = (\omega A)(\eta B) - (-1)^{(\deg \omega)(\deg \eta)}(\eta B)(\omega A). \tag{1.4}$$

The following vanishing result is then obvious.

Lemma 1.7 *For any A, $B \in \Omega^*(M; \mathrm{End}(E))$, the trace of $[A, B]$ vanishes.*

Lemma 1.8 *If ∇^E is a connection on E, then for any $A \in \Omega^*(M; \mathrm{End}(E))$, one has*

$$d \, \mathrm{tr}[A] = \mathrm{tr}\left[[\nabla^E, A]\right]. \tag{1.5}$$

Proof. First of all, if $\widetilde{\nabla}^E$ is another connection on E, then from the Leibniz rule in the definition of the connection, one verifies that

$$\nabla^E - \widetilde{\nabla}^E \in \Omega^1(M; \mathrm{End}(E)).$$

Thus by Lemma 1.7 one has

$$\mathrm{tr}\left[[\nabla^E - \widetilde{\nabla}^E, A]\right] = 0.$$

That is to say, the right hand side of (1.5) does not depend on the choice of ∇^E.

On the other hand, it is clear that the operations in the right hand side of (1.5) are local. Thus for any $x \in M$ one can choose a sufficiently small open

neighborhood U_x of x such that $E|_{U_x}$ is a trivial vector bundle. Then one can take a trivial connection on $E|_{U_x}$ for which (1.5) holds automatically.

By combining the above independence and local properties, one sees directly that (1.5) holds on the whole manifold M. □

Let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

be a power series in one variable.

Let R^E be the curvature of a connection ∇^E on E.

The trace of

$$f\left(R^E\right) = a_0 + a_1 R^E + \cdots + a_n \left(R^E\right)^n + \cdots$$

is an element in $\Omega^*(M)$.

We can now state a form of the **Chern-Weil theorem** (cf. [C]) as follows.

Theorem 1.9 *(i) The form* $\operatorname{tr}[f(R^E)]$ *is closed. That is,*

$$d \operatorname{tr}\left[f\left(R^E\right)\right] = 0;$$

(ii) If $\widetilde{\nabla}^E$ is another connection on E and $\widetilde{R}^E$ its curvature, then there is a differential form $\omega \in \Omega^*(M)$ such that

$$\operatorname{tr}\left[f\left(R^E\right)\right] - \operatorname{tr}\left[f\left(\widetilde{R}^E\right)\right] = d\omega. \tag{1.6}$$

Proof. (i) From Lemma 1.8 one verifies directly that

$$d \operatorname{tr}\left[f\left(R^E\right)\right] = \operatorname{tr}\left[[\nabla^E, f(R^E)]\right]$$

$$= \operatorname{tr}\left[a_1[\nabla^E, R^E] + \cdots + a_n[\nabla^E, (R^E)^n] + \cdots\right] = 0,$$

as for any integer $k \geq 0$ one has the obvious **Bianchi identity**

$$\left[\nabla^E, \left(R^E\right)^k\right] = \left[\nabla^E, \left(\nabla^E\right)^{2k}\right] = 0. \tag{1.7}$$

(ii) For any $t \in [0,1]$, let ∇_t^E be the deformed connection on E given by

$$\nabla_t^E = (1-t)\nabla^E + t\widetilde{\nabla}^E. \tag{1.8}$$

Then ∇^E_t is a connection on E such that $\nabla^E_0 = \nabla^E$ and $\nabla^E_1 = \widetilde{\nabla}^E$. Moreover,

$$\frac{d\nabla^E_t}{dt} = \widetilde{\nabla}^E - \nabla^E \in \Omega^1(M; \mathrm{End}(E)).$$

Let R^E_t, $t \in [0,1]$, denote the curvature of ∇^E_t. We study the change of $\mathrm{tr}[f(R^E_t)]$ when t changes in $[0,1]$.

Let $f'(x)$ be the power series obtained from the derivative of $f(x)$ with respect to x.

We deduce that

$$\frac{d}{dt}\mathrm{tr}\left[f\left(R^E_t\right)\right] = \mathrm{tr}\left[\frac{dR^E_t}{dt}f'\left(R^E_t\right)\right] = \mathrm{tr}\left[\frac{d\left(\nabla^E_t\right)^2}{dt}f'\left(R^E_t\right)\right]$$

$$= \mathrm{tr}\left[\left[\nabla^E_t, \frac{d\nabla^E_t}{dt}\right]f'\left(R^E_t\right)\right] = \mathrm{tr}\left[\left[\nabla^E_t, \frac{d\nabla^E_t}{dt}f'\left(R^E_t\right)\right]\right],$$

where the last equality follows from the Bianchi identity (1.7).

Combining with Lemma 1.8, one then gets

$$\frac{d}{dt}\mathrm{tr}\left[f\left(R^E_t\right)\right] = d\,\mathrm{tr}\left[\frac{d\nabla^E_t}{dt}f'\left(R^E_t\right)\right], \tag{1.9}$$

from which one gets

$$\mathrm{tr}\left[f\left(R^E\right)\right] - \mathrm{tr}\left[f\left(\widetilde{R}^E\right)\right] = -d\int_0^1 \mathrm{tr}\left[\frac{d\nabla^E_t}{dt}f'\left(R^E_t\right)\right]dt. \tag{1.10}$$

This completes the proof of the part (ii). □

1.5 Characteristic Forms, Classes and Numbers

By Theorem 1.9(i), $\mathrm{tr}[f(\frac{\sqrt{-1}}{2\pi}R^E)]$ is a closed differential form which determines a cohomology class $\left[\mathrm{tr}[f(\frac{\sqrt{-1}}{2\pi}R^E)]\right] \in H^*_{\mathrm{dR}}(M; \mathbf{C})$. While (1.6) says that this class does not depend on the choice of the connection ∇^E

Definition 1.10 (i) We call the differential form $\mathrm{tr}[f(\frac{\sqrt{-1}}{2\pi}R^E)]$ the **characteristic form** of E associated to ∇^E and f, and denote it by $f(E, \nabla^E)$.

(ii) We call the cohomology class $\left[\mathrm{tr}[f(\frac{\sqrt{-1}}{2\pi}R^E)]\right]$ the **characteristic class** of E associated to f, and denote it by $f(E)$.

Thus, a characteristic form is a differential form representative of the corresponding characteristic class. We also call a product of characteristic forms (classes) a (new) characteristic form (class).

We now assume M is oriented, so that one can integrate differential forms on M.

Let $E_1, \dots, E_k$ be k complex vector bundles over M, and $\nabla^{E_1}, \dots, \nabla^{E_k}$ the connections on them respectively.

Given k power series $f_1, \dots, f_k$, one can then form the characteristic form

$$f_1\left(E_1, \nabla^{E_1}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right) \in \Omega^*(M).$$

Let $\{f_1(E_1, \nabla^{E_1}) \cdots f_k(E_k, \nabla^{E_k})\}^{\max}$ denote its component in $\Omega^{\dim M}(M)$.

Lemma 1.11 *The number defined by*

$$\int_M f_1\left(E_1, \nabla^{E_1}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)$$

$$= \int_M \left\{f_1\left(E_1, \nabla^{E_1}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)\right\}^{\max} \tag{1.11}$$

does not independent on the choices of the connections ∇^{E_i}, $1 \leq i \leq k$.

Proof. Without loss of generality we assume that $\widetilde{\nabla}^{E_1}$ is another connection on E_1.

By Theorem 1.9(ii) there is a differential form ω on M such that

$$f_1\left(E_1, \nabla^{E_1}\right) - f_1\left(E_1, \widetilde{\nabla}^{E_1}\right) = d\omega.$$

One then uses Theorem 1.9(i) and the Stokes formula to deduce that

$$\int_M f_1\left(E_1, \nabla^{E_1}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)$$

$$- \int_M f_1\left(E_1, \widetilde{\nabla}^{E_1}\right) f_2\left(E_2, \nabla^{E_2}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)$$

$$= \int_M d\left(\omega f_2\left(E_2, \nabla^{E_2}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)\right) = 0,$$

from which the lemma follows easily. □

The number defined in (1.11) is called the **characteristic number** associated to the characteristic class $f_1(E_1) \cdots f_k(E_k)$, and is denoted by $\langle f_1(E_1) \cdots f_k(E_k), [M] \rangle$.

1.6 Some Examples

In this section we describe some well-known characteristic classes appearing in many places in geometry and topology.

1.6.1 *Chern Forms and Classes*

Let ∇^E be a connection on a complex vector bundle E over a smooth manifold M, and R^E the curvature of ∇^E.

The (total) **Chern form**, denoted by $c(E, \nabla^E)$, associated to ∇^E is defined by

$$c\left(E, \nabla^E\right) = \det\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right), \tag{1.12}$$

where I is the identity endomorphism of E.

Since

$$\det\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right) = \exp\left(\operatorname{tr}\left[\log\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right)\right]\right), \tag{1.13}$$

in view of the following power series expansion formulas for $\log(1+x)$ and $\exp(x)$,

$$\log(1+x) = x - \frac{x^2}{2} + \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots$$

and

$$\exp(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots,$$

one sees that $c(E, \nabla^E)$ is a characteristic form in the sense of Definition 1.10. The associated characteristic class, denoted by $c(E)$, is called the (total) **Chern class** of E.

By (1.12) it is clear that one has the decomposition of the (total) Chern form that

$$c\left(E, \nabla^E\right) = 1 + c_1\left(E, \nabla^E\right) + \cdots + c_k\left(E, \nabla^E\right) + \cdots$$

with each

$$c_i\left(E, \nabla^E\right) \in \Omega^{2i}(M).$$

We call $c_i(E, \nabla^E)$ the i-th **Chern form** associated to ∇^E, and its associated cohomology class, denoted by $c_i(E)$, the i-th **Chern class** of E.

Now if one rewrites (1.13) in the form that

$$\log\left(\det\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right)\right) = \operatorname{tr}\left[\log\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right)\right], \tag{1.14}$$

then from the above power series expansion formulas for $\log(1+x)$, one deduces that for any integer $k \geq 0$, $\operatorname{tr}[(R^E)^k]$ can be written as a linear combination of various products of $c_i(E, \nabla^E)$'s.

This establishes the fundamental importance of Chern classes in the theory of characteristic classes of complex vector bundles.

1.6.2 *Pontrjagin Classes for Real Vector Bundles*

Let now E be a *real* vector bundle over M, and ∇^E be a connection on E.* Let R^E be the curvature of ∇^E.

One sees easily that one can proceed in exactly the same way as in Sections 1.2-1.5 for real vector bundles with connections. Moreover, the Chern-Weil theorem can be formulated and proved in exactly the same way as in Theorem 1.9.

Now similar to the Chern forms for complex vector bundles, we define the (total) **Pontrjagin form** associated to ∇^E by

$$p\left(E, \nabla^E\right) = \det\left(\left(I - \left(\frac{R^E}{2\pi}\right)^2\right)^{1/2}\right) \tag{1.15}$$

*The definition of a connection on a real vector bundle is the same as that for a complex vector bundle in Definition 1.4, by simply replacing '**C**-linear' there to **R**-linear.

The associated characteristic class $p(E)$ is called the (total) **Pontrjagin class**.

Clearly, $p\left(E, \nabla^E\right)$ admits a decomposition

$$p\left(E, \nabla^E\right) = 1 + p_1\left(E, \nabla^E\right) + \cdots + p_k\left(E, \nabla^E\right) + \cdots$$

with each

$$p_i\left(E, \nabla^E\right) \in \Omega^{4i}(M).$$

We call $p_i\left(E, \nabla^E\right)$ the i-th **Pontrjagin form** associated to ∇^E, and call the associated class $p_i(E)$ the i-th **Pontrjagin class** of E.

The discussion at the end of Subsection 1.6.1 also applies here to show the fundamental importance of Pontrjagin classes in the theory of characteristic classes of real vector bundles.

Finally, if we denote by $E \otimes \mathbf{C}$ the complexification of E, then one has the following intimate relation between the Pontrjagin classes of E and the Chern classes of $E \otimes \mathbf{C}$, which is usually taken as the definition of Pontrjagin classes: for any integer $i \geq 0$,

$$p_i\left(E\right) = (-1)^i c_{2i}\left(E \otimes \mathbf{C}\right). \tag{1.16}$$

1.6.3 *Hirzebruch's L-class and $\widehat{A}$-class*

In this subsection we discuss some characteristic classes which are especially important when defined for the tangent bundle of a manifold. These classes were first defined by Hirzebruch (cf. [H]).

We start with the L-class, which is associated to the function

$$L(x) = \frac{x}{\tanh\left(x\right)}.$$

Let ∇^{TM} be a connection on the tangent vector bundle TM of a smooth closed manifold M. Let R^{TM} be the curvature of ∇^{TM}.

The L-form associated to ∇^{TM}, denoted by $L(TM, \nabla^{TM})$, is defined by

$$L\left(TM, \nabla^{TM}\right) = \det\left(\left(\frac{\frac{\sqrt{-1}}{2\pi}R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{2\pi}R^{TM}\right)}\right)^{1/2}\right) \in \Omega^*(M). \tag{1.17}$$

Its associated cohomology class, called the L-class of TM, is denoted by $L(TM)$.

As a very special case, if $\dim M = 4$, one has

$$\left\{L\left(TM,\nabla^{TM}\right)\right\}^{\max} = \frac{1}{3}p_1\left(TM,\nabla^{TM}\right). \qquad (1.18)$$

The importance of the L-class lies in the **Hirzebruch Signature theorem** (cf. [H]) which says that when M is oriented, then the L-**genus** of M, denoted by $L(M)$ and defined by

$$L(M) := \langle L(TM),[M]\rangle = \int_M L\left(TM,\nabla^{TM}\right),$$

equals to the **Signature**† of M. In particular, $L(M)$ is an *integer*.

The integrality of characteristic numbers such as $L(M)$ is highly non-trivial.

For one more example, we consider the $\widehat{A}$-class, which is associated to the function

$$\widehat{A}(x) = \frac{x/2}{\sinh(x/2)}.$$

Let M be as before a smooth compact oriented manifold. Let ∇^{TM} be a connection on TM. Let R^{TM} be the curvature of ∇^{TM}.

We define

$$\widehat{A}\left(TM,\nabla^{TM}\right) = \det\left(\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right)^{1/2}\right) \in \Omega^*(M), \qquad (1.19)$$

and denote the associated cohomology class by $\widehat{A}(TM)$.

As a special case, when $\dim M = 4$, one has

$$\left\{\widehat{A}\left(TM,\nabla^{TM}\right)\right\}^{\max} = -\frac{1}{24}p_1\left(TM,\nabla^{TM}\right). \qquad (1.20)$$

We define the $\widehat{A}$-genus of M, denoted by $\widehat{A}(M)$, by

$$\widehat{A}(M) = \left\langle \widehat{A}(TM),[M]\right\rangle = \int_M \widehat{A}\left(TM,\nabla^{TM}\right).$$

†The Signature of a manifold is defined as follows: if $\dim M = 4m$ for some integer m, then there is a natural symmetric quadratic form $H^{2m}_{\mathrm{dR}}(M;\mathbf{R}) \times H^{2m}_{\mathrm{dR}}(M;\mathbf{R}) \to \mathbf{R}$ defined by $([\omega],[\omega']) \to \int_M \omega\wedge\omega'$. Then the **Signature** of M is defined to be the signature of this quadratic form. If $\dim M$ is not divisible by 4, then define its Signature to be zero.

From (1.18) and (1.20), one sees that if $\dim M = 4$, then

$$L(M) = -8\widehat{A}(M). \tag{1.21}$$

Now by a theorem of Borel and Hirzebruch [BH], one knows that if M is *spin*,[‡] then $\widehat{A}(M)$ is still an *integer*. Moreover, when $\dim M \equiv 4 \bmod 8$, Atiyah and Hirzebruch [AH1] refined this result by showing that $\widehat{A}(M)$ is an *even* integer. Combining the later with (1.21), one recovers the famous Rokhlin theorem which says that the Signature of a smooth closed spin four manifold is divisible by 16.

The proofs in [AH1] and [BH] are purely topological and are indirect. The natural attempt to search for a more reasonable and direct explanation of these integrality results lead to the discovery of the celebrated Atiyah-Singer index theorem [AS]. We recommend the two excellent books of Berline-Getzler-Vergne [BGV] and Lawson-Michelsohn [LM] to the interested reader who wants to know more about index theory (The reader who knows Chinese can also consult the book of Yu [Y]).

On the other hand, there is a higher dimensional generalization of the above mentioned Rokhlin theorem due to Ochanine [O], which states that the Signature of a smooth closed $8k+4$ dimensional spin manifold is divisible by 16. We refer to the article of Liu [Liu] for a modern proof of this result. This proof involves elliptic genus and in particular a "miraculous cancellation" formula which generalizes (1.21) to arbitrarily dimensions and which in dimension 12 was first discovered by the physists Alvarez-Gaumé and Witten [AGW].

1.6.4 *K-groups and the Chern Character*

We now back to the case of complex vector bundles.

Still, let E be a complex vector bundle over a compact smooth manifold M. Let ∇^E be a ($\mathbf{C}$-linear) connection on E and let R^E denote its curvature.

The **Chern character form** associated to ∇^E is defined by

$$\mathrm{ch}\left(E, \nabla^E\right) = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi} R^E\right)\right] \in \Omega^{\mathrm{even}}(M). \tag{1.22}$$

The associated cohomology class, denoted by $\mathrm{ch}(E)$, is called the **Chern**

‡We refer to the book of Lawson-Michelsohn [LM] for more details about spin manifolds.

character of E.

The importance of the Chern character lies in its intimate relationships with the K-group of M.

Recall that if E, F are two complex vector bundles over M, then one can form the Whitney direct sum of E and F, denoted by $E \oplus F$, which is the vector bundle over M such that each fiber $(E \oplus F)_x$ at $x \in M$ is the direct sum $E_x \oplus F_x$ of the fibers E_x and F_x.

From (1.22), it is clear that if E and F are two complex vector bundles over M, then

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F) \quad \in \ H_{\mathrm{dR}}^{\mathrm{even}}(M; \mathbf{C}). \tag{1.23}$$

Denote by $\mathrm{Vect}(M)$ the set of all complex vector bundles over M, then under the Whitney direct sum operation, $\mathrm{Vect}(M)$ becomes a semi-abelian group.

We now introduce an equivalence relation '$\sim$' in $\mathrm{Vect}(M)$ as follows: two vector bundles E and F are equivalent to each other, if there exists a vector bundle G over M such that $E \oplus G$ is isomorphic to $F \oplus G$.

The quotient of $\mathrm{Vect}(M)$ by this equivalence relation, $\mathrm{Vect}(M)/\sim$, is still a semi-abelian group.

Following Atiyah and Hirzebruch [AH2], we define the K-group of M, denoted by $K(M)$, to be $\mathrm{Vect}(M)/\sim$, with the group structure canonically induced from the above semi-abelian group structure. Then by (1.23), one deduces easily that the Chern character can be extended naturally to a homomorphism

$$\mathrm{ch} : K(M) \longrightarrow H_{\mathrm{dR}}^{\mathrm{even}}(M; \mathbf{C}).$$

The importance of this homomorphism lies in the following result due to Atiyah and Hirzebruch [AH2], which says that if one ignores the torsion elements in $K(M)$, then the induced homomorphism

$$\mathrm{ch} : K(M) \otimes \mathbf{C} \longrightarrow H_{\mathrm{dR}}^{\mathrm{even}}(M; \mathbf{C})$$

is actually an *isomorphism.*

On the other hand, the integrality results in Subsection 1.6.3 can be generalized to allow complex vector bundles as coefficients. For example, one has that for any complex vector bundle E over an even dimensional

oriented spin closed manifold M, the characteristic number

$$\left\langle \widehat{A}(TM)\mathrm{ch}(E), [M] \right\rangle$$

is an integer (cf. [AH1]).

Of course, all these integrality results are special cases of the Atiyah-Singer index theorem [AS].

1.6.5 *The Chern-Simons Transgressed Form*

We now take a further look at the formula (1.10), which we rewrite as follows,

$$\mathrm{tr}\left[f\left(R^E\right)\right] - \mathrm{tr}\left[f\left(\widetilde{R}^E\right)\right] = -d\int_0^1 \mathrm{tr}\left[\frac{d\nabla_t^E}{dt} f'\left(R_t^E\right)\right] dt. \tag{1.24}$$

The transgressed term

$$-\int_0^1 \mathrm{tr}\left[\frac{d\nabla_t^E}{dt} f'\left(R_t^E\right)\right] dt \tag{1.25}$$

appearing in the right hand side is usually called a **Chern-Simons term.** In many interesting cases, it is a closed form and thus induces a cohomology class in $H^*_{\mathrm{dR}}(M;\mathbf{C})$.

A typical example is when both ∇^E and $\widetilde{\nabla}^E$ are *flat* connections, that is, when both R^E and $\widetilde{R}^E$ equal to zero.

We now examine another typical case, where $E = TM$, the tangent bundle of a smooth compact oriented *three* dimensional manifold M. [§]

Recall the standard result due originally to Stiefel (cf. [S]) that for a smooth compact oriented three manifold M, the tangent bundle TM is topologically trivial. Thus one can choose a fixed global basis e_1, e_2, e_3 of TM. Then every section $X \in \Gamma(TM)$ can be written as

$$X = f_1 e_1 + f_2 e_2 + f_3 e_3,$$

where f_1, f_2, f_3 are smooth functions on M.

Let d^{TM} denote the connection on TM defined by

$$d^{TM}(f_1 e_1 + f_2 e_2 + f_3 e_3) = df_1 \cdot e_1 + df_2 \cdot e_2 + df_3 \cdot e_3.$$

[§] As we have noted, although (1.10) is proved for complex vector bundles, the same strategy works without change also for the real vector bundles.

Then any connection ∇^{TM} on TM can be written as

$$\nabla^{TM} = d^{TM} + A$$

with

$$A \in \Omega^1(M; \mathrm{End}(TM)).$$

For any $t \in [0,1]$, set

$$\nabla_t^{TM} = d^{TM} + tA.$$

Take $f(x) = -x^2$.

By dimensional reason, the left hand side of (1.24) vanishes. Thus, the form in (1.25) is closed and one deduces that

$$-\int_0^1 \mathrm{tr}\left[\frac{d\nabla_t^{TM}}{dt} f'\left(R_t^{TM}\right)\right] dt = -\int_0^1 \mathrm{tr}\left[A(-2)\left(d^{TM} + tA\right)^2\right] dt$$

$$= 2\int_0^1 \mathrm{tr}\left[tA \wedge d^{TM}A + t^2 A \wedge A \wedge A\right] dt$$

$$= \mathrm{tr}\left[A \wedge d^{TM}A + \frac{2}{3} A \wedge A \wedge A\right],$$

which is precisely (up to a rescaling) the **Chern-Simons form** [CS] appearing recently in so many places in topology, geometry as well as in mathematical physics (cf. for example, the paper of Witten [W] on the Jones polynomial of knots).

1.7 Bott Vanishing Theorem for Foliations

As an application of the Chern-Weil theory, we discuss a vanishing theorem on foliations due to Bott. We recommend the interested reader to Volume 3 of Bott's Collected Papers [Bo] for further developments arising from this simple and beautiful result.

We will work on real vector bundles in this section.

1.7.1 *Foliations and the Bott Vanishing Theorem*

Let M be a closed manifold and TM its tangent vector bundle. Let $F \subset TM$ be a sub-vector bundle of TM. We say F is an *integrable* subbundle of TM if for any two smooth sections X, $Y \in \Gamma(F)$ of F, their Lie bracket is also a section of F, that is,

$$[X,Y] \in \Gamma(F). \tag{1.26}$$

If such an integrable subbundle $F \subset TM$ exists on M, then we call M a **foliation** (or a **foliated space**) foliated by F.

We now assume that M is a foliation which is foliated by an integrable subbundle F of TM. Let TM/F be the quotient vector bundle of TM by F.

Let $p_{i_1}(TM/F)$, ..., $p_{i_k}(TM/F)$ be k Pontrjagin classes of TM/F.

We can now state the **Bott vanishing theorem** as follows.

Theorem 1.12 *If* $i_1 + \cdots + i_k > (\dim M - \dim F)/2$, *then*

$$p_{i_1}(TM/F) \cdots p_{i_k}(TM/F) = 0 \quad \text{in} \quad H_{\mathrm{dR}}^{4(i_1+\cdots+i_k)}(M;\mathbf{R}) \tag{1.27}$$

Proof. To simplify the exposition, we take a Riemannian metric g^{TM} on TM.¶ Then TM admits an orthogonal decomposition

$$TM = F \oplus F^{\perp}$$

such that F and $F^{\perp}$ are orthogonal to each other with respect to g^{TM}. Moreover, TM/F can be identified with $F^{\perp}$.

Let ∇^{TM} be the Levi-Civita connection on TM associated to g^{TM}. Let g^F, $g^{F^\perp}$ be the metrics on F, $F^\perp$ induced from g^{TM}. Let p, $p^\perp$ denote the orthogonal projection from TM to F, $F^\perp$ respectively. Set

$$\nabla^F = p\nabla^{TM}p, \qquad \nabla^{F^\perp} = p^\perp \nabla^{TM} p^\perp.$$

Then one verifies easily that ∇^F, $\nabla^{F^\perp}$ are connections on F, $F^\perp$ respectively. Moreover, they preserve g^F, $g^{F^\perp}$ respectively.

It is clear that to prove (1.27) one needs only to show that there is a smooth form $\omega \in \Omega^*(M)$ such that when $i_1 + \cdots + i_k > (\dim M - \dim F)/2$,

$$p_{i_1}\left(F^\perp, \nabla^{F^\perp}\right) \cdots p_{i_k}\left(F^\perp, \nabla^{F^\perp}\right) = d\omega. \tag{1.28}$$

¶For the basics of Riemannian geometry, see the book of Chern-Chen-Lam [CCL].

Following Bott, we will construct a new connection $\widetilde{\nabla}^{F^\perp}$ on $F^\perp$ such that

$$p_{i_1}\left(F^\perp,\widetilde{\nabla}^{F^\perp}\right)\cdots p_{i_k}\left(F^\perp,\widetilde{\nabla}^{F^\perp}\right)=0, \tag{1.29}$$

when $i_1+\cdots+i_k>(\dim M-\dim F)/2$.

The **Bott connection** $\widetilde{\nabla}^{F^\perp}$ on $F^\perp$ can be defined as follows.

Definition 1.13 For any $X\in\Gamma(TM)$, $U\in\Gamma(F^\perp)$,

(i) If $X\in\Gamma(F)$, we define

$$\widetilde{\nabla}_X^{F^\perp}U=p^\perp[X,U];$$

(ii) If $X\in\Gamma(F^\perp)$, set $\widetilde{\nabla}_X^{F^\perp}U=\nabla_X^{F^\perp}U$.

The part (ii) is not essential. The importance of the part (i) lies in the following result of Bott.

Let $\widetilde{R}^{F^\perp}$ denote the curvature of $\widetilde{\nabla}^{F^\perp}$

Lemma 1.14 *For any X, $Y\in\Gamma(F)$, one has*

$$\widetilde{R}^{F^\perp}(X,Y)=0.$$

Proof. Let $Z\in\Gamma(F^\perp)$ be any smooth section of $F^\perp$. By (1.3) and (i) above,

$$\widetilde{R}^{F^\perp}(X,Y)Z=\widetilde{\nabla}_X^{F^\perp}\widetilde{\nabla}_Y^{F^\perp}Z-\widetilde{\nabla}_Y^{F^\perp}\widetilde{\nabla}_X^{F^\perp}Z-\widetilde{\nabla}_{[X,Y]}^{F^\perp}Z$$

$$=\widetilde{\nabla}_X^{F^\perp}p^\perp[Y,Z]-\widetilde{\nabla}_Y^{F^\perp}p^\perp[X,Z]-p^\perp[[X,Y],Z]$$

$$=p^\perp\left([X,p^\perp[Y,Z]]+[Y,p^\perp[Z,X]]+[Z,[X,Y]]\right)$$

$$=p^\perp\left([X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]\right)$$

$$-p^\perp[X,p[Y,Z]]-p^\perp[Y,p[Z,X]]=0,$$

where the last equality follows from (1.26) and the Jacobi identity.

This completes the proof of Lemma 1.14. □

Let $F^{\perp,*}$ denote the dual bundle of $F^\perp$.

From Lemma 1.14, one sees easily that

$$\widetilde{R}^{F^\perp} \in \Gamma\left(F^{\perp,*}\right) \wedge \Omega^*\left(M; \mathrm{End}\left(F^\perp\right)\right).$$

Thus, for any integer j with $1 \le j \le k$,

$$p_{i_j}\left(F^\perp, \widetilde{\nabla}^{F^\perp}\right) \in \Gamma\left(\Lambda^{2i_j}\left(F^{\perp,*}\right)\right) \wedge \Omega^*(M). \tag{1.30}$$

From (1.30), one deduces that

$$p_{i_1}\left(F^\perp, \widetilde{\nabla}^{F^\perp}\right) \cdots p_{i_k}\left(F^\perp, \widetilde{\nabla}^{F^\perp}\right) \in \Gamma\left(\Lambda^{2(i_1+\cdots+i_k)}\left(F^{\perp,*}\right)\right) \wedge \Omega^*(M). \tag{1.31}$$

Since $\dim F^\perp = \dim M - \dim F$, one sees directly from (1.31) that when $i_1 + \cdots + i_k > (\dim M - \dim F)/2$, formula (1.29) holds.

From (1.29) and the Chern-Weil theorem, one gets (1.28).

The proof of Theorem 1.12 is thus completed. □

1.7.2 *Adiabatic Limit and the Bott Connection*

One may argue that from the geometric point of view, the connection $\nabla^{F^\perp}$ is also a natural connection on $F^\perp$. In fact, by passing g^{TM} to its *adiabatic limit*, one sees that the underlying limit of $\nabla^{F^\perp}$ and the Bott connection $\widetilde{\nabla}^{F^\perp}$ are ultimately related.

To be more precise, for any $\varepsilon > 0$, let g_ε^{TM} be the metric on TM defined by

$$g^{TM,\varepsilon} = g^F \oplus \frac{1}{\varepsilon} g^{F^\perp}$$

Let $\nabla^{TM,\varepsilon}$ be the Levi-Civita connection of $g^{TM,\varepsilon}$. Let $\nabla^{F,\varepsilon}$ (resp. $\nabla^{F^\perp,\varepsilon}$) be the restriction of $\nabla^{TM,\varepsilon}$ to F (resp. $F^\perp$).

We will examine the behavior of $\nabla^{F^\perp,\varepsilon}$ as $\varepsilon \to 0$.

The process of taking the limit $\varepsilon \to 0$ is called taking the *adiabatic limit.*

The standard formula for Levi-Civita connection (cf. [CCL]) implies that for any $X \in \Gamma(F)$, U, $V \in \Gamma(F^\perp)$,

$$\left\langle \nabla_X^{F^\perp,\varepsilon} U, V \right\rangle = \langle [X,U], V\rangle - \frac{1}{2}\left\langle X, \nabla_V^{TM} U + \nabla_U^{TM} V\right\rangle - \frac{\varepsilon}{2}\langle X, [U,V]\rangle. \tag{1.32}$$

Let $\widetilde{\nabla}^{F^\perp,*}$ be the connection on $F^\perp$ which is dual to $\widetilde{\nabla}^{F^\perp}$. That is, for any sections U, $V \in \Gamma(F^\perp)$,

$$d\langle U,V\rangle = \left\langle \widetilde{\nabla}^{F^\perp}U, V\right\rangle + \left\langle U, \widetilde{\nabla}^{F^\perp,*}V\right\rangle. \tag{1.33}$$

Set

$$\omega^{F^\perp} = \widetilde{\nabla}^{F^\perp,*} - \widetilde{\nabla}^{F^\perp}. \tag{1.34}$$

Let $\widehat{\nabla}^{F^\perp}$ be the naturally induced connection on $F^\perp$ defined by

$$\widehat{\nabla}^{F^\perp} = \widetilde{\nabla}^{F^\perp} + \frac{\omega^{F^\perp}}{2}. \tag{1.35}$$

Then one verifies easily that $\widehat{\nabla}^{F^\perp}$ preserves $g^{F^\perp}$.

The following result is taken from [LiuZ].

Theorem 1.15 *For any smooth section $X \in \Gamma(F)$, one has,*

$$\lim_{\varepsilon\to 0} \nabla_X^{F^\perp,\varepsilon} = \widehat{\nabla}_X^{F^\perp}. \tag{1.36}$$

Proof. For any $X \in \Gamma(F)$, U, $V \in \Gamma(F^\perp)$, by (1.33) and (1.34) one has

$$\begin{aligned}\omega^{F^\perp}(X)(U,V) &= -\left\langle U, \widetilde{\nabla}_X^{F^\perp}V\right\rangle - \left\langle \widetilde{\nabla}_X^{F^\perp}U, V\right\rangle + X\langle U,V\rangle \\ &= -\langle U,[X,V]\rangle - \langle [X,U],V\rangle + X\langle U,V\rangle \\ &= -\left\langle U, \nabla_X^{TM}V - \nabla_V^{TM}X\right\rangle - \left\langle \nabla_X^{TM}U - \nabla_U^{TM}X, V\right\rangle + X\langle U,V\rangle \\ &= -\left\langle \nabla_U^{TM}V, X\right\rangle - \left\langle \nabla_V^{TM}U, X\right\rangle - \left\langle U, \nabla_X^{TM}V\right\rangle - \left\langle \nabla_X^{TM}U, V\right\rangle + X\langle U,V\rangle.\end{aligned} \tag{1.37}$$

Note that the last three terms cancel. So (1.36) follows directly from (1.32), (1.35) and (1.37). □

Remark 1.16 If for any $X \in \Gamma(F)$, $\omega^{F^\perp}(X) = 0$, then one says that $(M, F, g^{F^\perp})$ admits a **Riemannian foliation** structure (cf. [T]).

1.8 Chern-Weil Theory in Odd Dimension

The theory of characteristic forms and classes we have discussed in the previous sections are mainly even dimensional. In this section, we will describe an odd dimensional analogue of this theory.

Let M be a smooth closed manifold. Let g be a smooth map from M to the general linear group $GL(N, \mathbf{C})$ with $N > 0$ a positive integer:

$$g : M \longrightarrow GL(N, \mathbf{C}). \tag{1.38}$$

Let $\mathbf{C}^N|_M$ denote the trivial complex vector bundle of rank N over M. Then the above element g can also be viewed as a section of $\mathrm{Aut}(\mathbf{C}^N|_M)$.

Let d denote a trivial connection on $\mathbf{C}^N|_M$. Then one gets a natural element

$$g^{-1}dg \in \Omega^1\left(M; \mathrm{End}\left(\mathbf{C}^N|_M\right)\right). \tag{1.39}$$

If n is a positive even integer, one verifies that

$$\mathrm{tr}\left[\left(g^{-1}dg\right)^n\right] = \frac{1}{2}\mathrm{tr}\left[\left[\left(g^{-1}dg\right)^{n-1}, g^{-1}dg\right]\right] = 0. \tag{1.40}$$

On the other hand, from the equality $gg^{-1} = 1$, one deduces that

$$dg^{-1} = -g^{-1}(dg)g^{-1}. \tag{1.41}$$

From (1.40) and (1.41) one deduces that if n is a positive odd integer, then

$$d\,\mathrm{tr}\left[\left(g^{-1}dg\right)^n\right] = n\mathrm{tr}\left[d\left(g^{-1}dg\right)\left(g^{-1}dg\right)^{n-1}\right]$$

$$= -n\mathrm{tr}\left[\left(g^{-1}dg\right)^{n+1}\right] = 0. \tag{1.42}$$

The following lemma shows that the cohomology class determined by the closed form $\mathrm{tr}[(g^{-1}dg)^n]$ does not depend on smooth deformations of $g : M \to GL(N, \mathbf{C})$.

Lemma 1.17 *If $g_t : M \to GL(N, \mathbf{C})$ depends smoothly on $t \in [0, 1]$, then for any positive odd integer n, the following identity holds,*

$$\frac{\partial}{\partial t}\mathrm{tr}\left[\left(g_t^{-1}dg_t\right)^n\right] = nd\,\mathrm{tr}\left[g_t^{-1}\frac{\partial g_t}{\partial t}\left(g_t^{-1}dg_t\right)^{n-1}\right]. \tag{1.43}$$

Proof. By an analogue of (1.41), one deduces that

$$\frac{\partial}{\partial t}\left(g_t^{-1}dg_t\right) = \frac{\partial g_t^{-1}}{\partial t}dg_t + g_t^{-1}d\frac{\partial g_t}{\partial t}$$

$$= -\left(g_t^{-1}\frac{\partial g_t}{\partial t}\right)g_t^{-1}dg_t + g_t^{-1}d\frac{\partial g_t}{\partial t}$$

$$= -\left(g_t^{-1}\frac{\partial g_t}{\partial t}\right)g_t^{-1}dg_t + \left(g_t^{-1}dg_t\right)\left(g_t^{-1}\frac{\partial g_t}{\partial t}\right) + d\left(g_t^{-1}\frac{\partial g_t}{\partial t}\right) \tag{1.44}$$

One also verifies that

$$d\left(g_t^{-1}dg_t\right)^2 = d\left(g_t^{-1}dg_t\right)g_t^{-1}dg_t - g_t^{-1}dg_t d\left(g_t^{-1}dg_t\right) = 0,$$

from which one deduces that for any positive even integer k,

$$d\left(g_t^{-1}dg_t\right)^k = 0. \tag{1.45}$$

From (1.40), (1.44) and (1.45), one verifies that

$$\frac{\partial}{\partial t}\mathrm{tr}\left[\left(g_t^{-1}dg_t\right)^n\right] = n\mathrm{tr}\left[\frac{\partial}{\partial t}\left(g_t^{-1}dg_t\right)\left(g_t^{-1}dg_t\right)^{n-1}\right]$$

$$= n\mathrm{tr}\left[\left[g_t^{-1}dg_t, g_t^{-1}\frac{\partial g_t}{\partial t}\right]\left(g_t^{-1}dg_t\right)^{n-1}\right] + n\mathrm{tr}\left[d\left(g_t^{-1}\frac{\partial g_t}{\partial t}\right)\left(g_t^{-1}dg_t\right)^{n-1}\right]$$

$$= n\mathrm{tr}\left[\left[g_t^{-1}dg_t, g_t^{-1}\frac{\partial g_t}{\partial t}\left(g_t^{-1}dg_t\right)^{n-1}\right]\right] + n\mathrm{tr}\left[d\left(g_t^{-1}\frac{\partial g_t}{\partial t}\left(g_t^{-1}dg_t\right)^{n-1}\right)\right]$$

$$= nd\,\mathrm{tr}\left[g_t^{-1}\frac{\partial g_t}{\partial t}\left(g_t^{-1}dg_t\right)^{n-1}\right].$$

The proof of Lemma 1.17 is completed. □

Corollary 1.18 *If $f,\ g : M \to GL(N, \mathbf{C})$ are two smooth maps from M to the general linear group $GL(N, \mathbf{C})$, then for any positive odd integer n, there exists $\omega_n \in \Omega^{n-1}(M)$ such that the following transgression formula holds,*

$$\mathrm{tr}\left[\left((fg)^{-1}d(fg)\right)^n\right] = \mathrm{tr}\left[\left(f^{-1}df\right)^n\right] + \mathrm{tr}\left[\left(g^{-1}dg\right)^n\right] + d\,\omega_n. \tag{1.46}$$

Proof. We consider the direct sum of two trivial complex vector bundles

$$\mathbf{C}^{2N}\big|_M = \mathbf{C}^N\big|_M \oplus \mathbf{C}^N\big|_M .$$

We equip $\mathbf{C}^{2N}|_M$ with the trivial connection induced from that on $\mathbf{C}^N$.

For any $u \in [0, \frac{\pi}{2}]$, let $h(u) : M \to GL(2N, \mathbf{C})$ be defined by

$$h(u) = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix}.$$

Clearly,

$$h(0) = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \quad \text{and} \quad h\left(\frac{\pi}{2}\right) = \begin{pmatrix} fg & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $h(u)$ provides a smooth deformation between two sections $(fg, 1)$ and (f, g) in $\Gamma(\mathrm{Aut}(\mathbf{C}^{2N}|_M))$.

By applying Lemma 1.17 to $h(u)$, one gets (1.46). □

Corollary 1.19 *Let $g \in \Gamma(\mathrm{Aut}(\mathbf{C}^N|_M))$. If d' is another trivial connection on $\mathbf{C}^N|_M$, then for any positive odd integer n, there exists $\omega_n \in \Omega^{n-1}(M)$ such that the following transgression formula holds,*

$$\operatorname{tr}\left[\left(g^{-1}dg\right)^n\right] = \operatorname{tr}\left[\left(g^{-1}d'g\right)^n\right] + d\,\omega_n. \tag{1.47}$$

Proof. Clearly, there exists $A \in \Gamma(\mathrm{Aut}(\mathbf{C}^N|_M))$ such that

$$d' = A^{-1} \cdot d \cdot A. \tag{1.48}$$

From (1.48), one deduces that

$$g^{-1}d'g = g^{-1} \cdot d' \cdot g - d'$$

$$= g^{-1} \cdot A^{-1} \cdot d \cdot A \cdot g - A^{-1} \cdot d \cdot A$$

$$= A^{-1}\left(A \cdot g^{-1} \cdot A^{-1} \cdot d \cdot A \cdot g \cdot A^{-1} - d\right) A$$

$$= A^{-1}\left(\left(AgA^{-1}\right)^{-1} d\left(AgA^{-1}\right)\right) A. \tag{1.49}$$

From (1.41), (1.49) and Lemma 1.17, one sees that for any positive odd integer n, there exists $\omega_n \in \Omega^{n-1}(M)$ such that

$$\operatorname{tr}\left[\left(g^{-1}d'g\right)^n\right] = \operatorname{tr}\left[\left(A^{-1}\left(\left(AgA^{-1}\right)^{-1}d\left(AgA^{-1}\right)\right)A\right)^n\right]$$

$$= \operatorname{tr}\left[\left(AdA^{-1}\right)^n\right] + \operatorname{tr}\left[\left(g^{-1}dg\right)^n\right] + \operatorname{tr}\left[\left(A^{-1}dA\right)^n\right] - d\,\omega_n$$

$$= \operatorname{tr}\left[\left(g^{-1}dg\right)^n\right] - d\,\omega_n,$$

which is exactly (1.47). □

Remark 1.20 By Lemma 1.17 and Corollary 1.19, one sees that the cohomology class determined by $\operatorname{tr}[(g^{-1}dg)^n]$ depends only on the homotopy class of $g : M \to GL(N, \mathbf{C})$.

Let n be a positive odd integer, we call the closed n-form

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{n+1}{2}} \operatorname{tr}\left[\left(g^{-1}dg\right)^n\right]$$

the n-th **Chern form** associated to g, d and denote it by $c_n(g, d)$. The associated cohomology class will be called the n-th **Chern class** associated to the homotopy class $[g]$ of g. We denote this class by $c_n([g])$.

We define the **odd Chern character form** associated to g and d by

$$\operatorname{ch}(g, d) = \sum_{n=0}^{+\infty} \frac{n!}{(2n+1)!} c_{2n+1}(g, d). \tag{1.50}$$

Let $\operatorname{ch}([g])$ denote the associated cohomology class which we call the **odd Chern character** associated to $[g]$.

For any two f, $g : M \to GL(N, \mathbf{C})$, by Corollary 1.18, one has the following additive property,

$$\operatorname{ch}([fg]) = \operatorname{ch}([f]) + \operatorname{ch}([g]) \quad \text{in} \quad H_{\mathrm{dR}}^{\mathrm{odd}}(M).$$

The following integrality result partly explains the choice of coefficients in (1.50): if M is an odd dimensional closed oriented spin manifold, E a

complex vector bundle over M and $g : M \to GL(N, \mathbf{C})$ a smooth map from M to the general linear group $GL(N, \mathbf{C})$, then

$$\left\langle \widehat{A}\,(TM)\, \mathrm{ch}(E)\mathrm{ch}([g]), [M] \right\rangle$$

is an integer.

We refer to Baum-Douglas [BD] and Getzler [G] for the index theoretic interpretations of this integrality result.

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Chapter 2

Bott and Duistermaat-Heckman Formulas

In Chapter One we have defined characteristic classes and numbers in terms of curvatures of connections on vector bundles. A natural question is how to compute these characteristic numbers. In this chapter we will discuss a localization formula due to Bott [Bo] which shows that for a compact manifold admitting a compact Lie group action, the calculation of characteristic numbers on this manifold can be reduced to the fixed point set of the group action.

The philosophy of localizing a computation on a manifold to that on the fixed point set of certain group actions on that manifold has a wide range of implications in topology and geometry. The Duistermaat-Heckman formula [DH] in symplectic geometry is another important example for this.

It turns out that the Bott localization formula and the Duistermaat-Heckman formula can be put into the unified framework of the equivariant cohomology theory.

In this chapter, we will first prove an equivariant localization formula due to Berline-Vergne [BV] and Atiyah-Bott [AB], then show how the Bott and Duistermaat-Heckman formulas can be deduced from it.

2.1 Berline-Vergne Localization Formula

Let M be an even dimensional smooth closed oriented manifold. We assume that M admits an S^1-action.

Let g^{TM} be a Riemannian metric on TM, the tangent vector bundle of

M. Without loss of generality we assume that g^{TM} is S^1-invariant.*

The S^1-action on M induces an action on $C^\infty(M)$ such that for any $f \in C^\infty(M)$, $x \in M$ and $g \in S^1$, $(g \cdot f)(x) = f(xg)$.

Let $\mathbf{t} \in \mathrm{Lie}(S^1)$ be a generator of the Lie algebra of S^1. Then $\mathbf{t}$ induces canonically a vector field K in the following manner: for any $f \in C^\infty(M)$ and $x \in M$,

$$(Kf)(x) = \frac{d}{d\varepsilon} f(x \exp(\varepsilon \mathbf{t}))\Big|_{\varepsilon=0}.$$

Since the S^1-action preserves g^{TM}, K is a **Killing** vector field on M. It induces a *skew-adjoint* homomorphism from TM to TM by $X \mapsto \nabla_X^{TM} K$, where ∇^{TM} is the Levi-Civita connection associated to g^{TM}. That is, for any X, $Y \in \Gamma(TM)$, one has

$$\left\langle \nabla_X^{TM} K, Y \right\rangle + \left\langle \nabla_Y^{TM} K, X \right\rangle = 0. \tag{2.1}$$

Proof of (2.1). Let $\mathcal{L}_K$ denote the Lie derivative of K on $\Gamma(TM)$. Since the S^1-action preserves g^{TM}, $\mathcal{L}_K$ also preserves g^{TM}. That is, for any X, $Y \in \Gamma(TM)$, one has

$$\begin{aligned} K\langle X, Y\rangle &= \langle \mathcal{L}_K X, Y\rangle + \langle X, \mathcal{L}_K Y\rangle \\ &= \langle [K, X], Y\rangle + \langle X, [K, Y]\rangle \\ &= \left\langle \nabla_K^{TM} X - \nabla_X^{TM} K, Y \right\rangle + \left\langle \nabla_K^{TM} Y - \nabla_Y^{TM} K, X \right\rangle \\ &= K\langle X, Y\rangle - \left\langle \nabla_X^{TM} K, Y \right\rangle - \left\langle \nabla_Y^{TM} K, X \right\rangle, \end{aligned}$$

from which (2.1) follows. □

The Lie derivative $\mathcal{L}_K$ on $\Gamma(TM)$ induces canonically an action on $\Omega^*(M)$ which we still denote by $\mathcal{L}_K$ and call it the Lie derivative of K on $\Omega^*(M)$. The following *Cartan homotopy formula* on $\Omega^*(M)$ is well-known,

$$\mathcal{L}_K = d i_K + i_K d, \tag{2.2}$$

where $i_K : \Omega^*(M) \to \Omega^{*-1}(M)$ is the interior multiplication induced by the contraction of K.

*In fact, given any metric on TM, one can integrate it over S^1 to get an S^1-invariant one.

Proof of (2.2). First of all, for any $f \in C^\infty(M)$, one verifies directly that

$$\mathcal{L}_K f = (di_K + i_K d)\, f = Kf.$$

Secondly, since $\mathcal{L}_K$ commutes with the exterior differential d, one has

$$\mathcal{L}_K df = d\mathcal{L}_K f = di_K df = (di_K + i_K d)df.$$

Since locally every one form can be written in the form df for some $f \in C^\infty(M)$, one sees that (2.2) also verifies for all one forms on M.

Finally, since both sides of (2.2) verify the Leibnize rule, from the above two facts one sees by induction that (2.2) holds for all forms on M. □

Let

$$\Omega_K^*(M) = \{\omega \in \Omega^*(M) : \mathcal{L}_K \omega = 0\}$$

be the subspace of $\mathcal{L}_K$-**invariant forms.**

Set

$$d_K = d + i_K : \Omega^*(M) \to \Omega^*(M). \tag{2.3}$$

One verifies easily that

$$d_K^2 = di_K + i_K d = \mathcal{L}_K. \tag{2.4}$$

Thus d_K preserves $\Omega_K^*(M)$ and

$$d_K^2|_{\Omega_K^*(M)} = 0.$$

The corresponding cohomology group

$$H_K^*(M) = \frac{\ker d_K|_{\Omega_K^*(M)}}{\operatorname{Im} d_K|_{\Omega_K^*(M)}}$$

is called the S^1 **equivariant cohomology** of M.

Consider now any element $\omega \in \Omega^*(M)$. We say ω is d_K-**closed** if $d_K\omega = 0$. By (2.4), a d_K-closed form is $\mathcal{L}_K$-invariant.

The equivariant localization formula due to Berline-Vergne [BV] (see also Atiyah-Bott [AB]) shows that the integration of a d_K-closed differential form over M can be localized to the zero set of the Killing vector field K. For simplicity, we will only prove this formula for the special case where the zero set of K is discrete.

We start with the simplest case.

Proposition 2.1 *If K has no zeros on M, then for any $\omega \in \Omega^*(M)$ which is d_K-closed, one has $\int_M \omega = 0$.*

Proof. We use a method due to Bismut [Bi2].

Let $\theta \in \Omega^1(M)$ be the one form on M such that for any $X \in \Gamma(TM)$,

$$i_X\theta = \langle X, K\rangle.$$

Since $\mathcal{L}_K$ preserves g^{TM}, one verifies easily that

$$\mathcal{L}_K\theta = 0.$$

From (2.4), one then sees that $(d + i_K)\theta$ is d_K-closed.

The following lemma is due to Bismut [Bi2].

Lemma 2.2 *For any $T \geq 0$, one has*

$$\int_M \omega = \int_M \omega \exp\left(-Td_K\theta\right). \tag{2.5}$$

Proof. Since $d_K\theta$ is d_K-closed, one verifies directly that

$$\exp\left(-Td_K\theta\right) - 1 = d_K\left(\sum_{i=1}^{+\infty} \frac{(-1)^i}{i!} T^i\theta \wedge (d_K\theta)^{i-1}\right),$$

from which one verifies directly, as $d_K\omega = 0$, that

$$\int_M \omega \exp\left(-Td_K\theta\right) - \int_M \omega$$

$$= (-1)^{\deg\omega}\int_M d_K\left(\omega\sum_{i=1}^{+\infty}\frac{(-1)^i}{i!}T^i\theta\wedge(d_K\theta)^{i-1}\right)$$

$$= 0.$$

□

Since

$$d_K\theta = d\theta + i_K\theta = d\theta + |K|^2,$$

one sees that

$$\int_M \omega \exp\left(-T d_K \theta\right)$$

$$= \int_M \omega \exp\left(-T|K|^2\right) \left(\sum_{i=0}^{\dim M/2} \frac{(-1)^i}{i!} T^i \left(d\theta\right)^i \right). \tag{2.6}$$

Now as K has no zeros on M, $|K|$ has a positive lower bound $\delta > 0$ on M. One then sees easily that when $T \to +\infty$, the right hand side of (2.6) is of exponential decay. Combining this fact with Lemma 2.2, one gets Proposition 2.1. □

We now assume that the zero set of K, which we denote by zero(K), is discrete.

By using the exponential map at every $p \in \text{zero}(K)$, one can assume that for every point p in the zero set of K, there is a sufficiently small open neighborhood U_p of p and an oriented coordinate system $(x^1, \ldots, x^{2l})$ with $l = \frac{1}{2}\dim M$ such that on U_p one has

$$g^{TM} = \left(dx^1\right)^2 + \cdots + \left(dx^{2l}\right)^2$$

and

$$K = \lambda_1 \left(x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} \right) + \cdots + \lambda_l \left(x^{2l} \frac{\partial}{\partial x^{2l-1}} - x^{2l-1} \frac{\partial}{\partial x^{2l}} \right)$$

with each $\lambda_i \neq 0$ for $1 \leq i \leq l$.

Set

$$\lambda(p) = \lambda_1 \cdots \lambda_l.$$

We now state the Berline-Vergne localization formula [BV] in this case as follows.

Theorem 2.3 *If the zero set of K is discrete, then for any d_K-closed differential form $\omega \in \Omega^*(M)$, one has*

$$\int_M \omega = (2\pi)^l \sum_{p \in \text{zero}(K)} \frac{\omega^{[0]}(p)}{\lambda(p)}, \tag{2.7}$$

where $\omega^{[0]} \in C^\infty(M)$ is the 0-th degree component of ω.

Proof. By (2.5), one has

$$\int_M \omega = \int_{M\setminus \cup_{p\in \mathrm{zero}(K)} U_p} \omega \exp\left(-T d_K \theta\right) + \sum_{p\in \mathrm{zero}(K)} \int_{U_p} \omega \exp\left(-T d_K\theta\right). \tag{2.8}$$

Since K has no zeros on $M \setminus \cup_{p\in\mathrm{zero}(K)} U_p$, one can proceed as in the proof of Proposition 2.1 to show that

$$\int_{M\setminus \cup_{p\in \mathrm{zero}(K)} U_p} \omega \exp\left(-T d_K\theta\right) \to 0 \quad \text{as} \quad T \to +\infty. \tag{2.9}$$

Now on each U_p, one verifies directly that

$$\theta = \lambda_1 \left(x^2 dx^1 - x^1 dx^2\right) + \cdots + \lambda_l \left(x^{2l} dx^{2l-1} - x^{2l-1} dx^{2l}\right).$$

Thus,

$$d\theta = -2\left(\lambda_1 dx^1 dx^2 + \cdots + \lambda_l dx^{2l-1} dx^{2l}\right). \tag{2.10}$$

Also, one verifies directly that, on U_p,

$$|K|^2 = \lambda_1^2 \left(\left(x^1\right)^2 + \left(x^2\right)^2\right) + \cdots + \lambda_l^2 \left(\left(x^{2l-1}\right)^2 + \left(x^{2l}\right)^2\right). \tag{2.11}$$

For any integer i such that $0 \le i \le 2l = \dim M$, let $\omega^{[i]} \in \Omega^i(M)$ denote the corresponding component of ω, then one verifies directly that for any $p \in \mathrm{zero}(K)$,

$$\int_{U_p} \omega \exp\left(-T d_K \theta\right) = \sum_{i=0}^{l} \frac{(-1)^i}{i!} \int_{U_p} \omega^{[2l-2i]} \exp\left(-T|K|^2\right) \left(T^i \left(d\theta\right)^i\right).$$

Now we make the rescaling change of the coordinate system

$$x = \left(x^1, \ldots, x^{2l}\right) \to \sqrt{T} x = \left(\sqrt{T} x^1, \ldots, \sqrt{T} x^{2l}\right).$$

By (2.10) and (2.11), one finds that if $0 \le i \le l-1$, then

$$\int_{U_p} \omega^{[2l-2i]} \exp\left(-T|K|^2\right) \left(T^i \left(d\theta\right)^i\right)$$

$$= \int_{\sqrt{T} U_p} \left(\frac{1}{T}\right)^{l-i} \omega^{[2l-2i]} \left(\frac{x}{\sqrt{T}}\right) \exp\left(-|K|^2\right) \left(d\theta\right)^i$$

$$\longrightarrow 0 \quad \text{as} \quad T \to +\infty. \tag{2.12}$$

On the other hand, if $i = l$, then one computes that

$$\frac{(-1)^i}{i!}\int_{U_p}\omega^{[2l-2i]}\exp\left(-T|K|^2\right)\left(T^i\,(d\theta)^i\right) = \int_{\sqrt{T}U_p}\omega^{[0]}\left(\frac{x}{\sqrt{T}}\right)$$

$$\cdot\exp\left(-\left(\lambda_1^2\left(\left(x^1\right)^2+\left(x^2\right)^2\right)+\cdots+\lambda_l^2\left(\left(x^{2l-1}\right)^2+\left(x^{2l}\right)^2\right)\right)\right)$$

$$\cdot\, 2^l\lambda_1\cdots\lambda_l dx^1\cdots dx^{2l}$$

$$\longrightarrow (2\pi)^l\frac{\omega^{[0]}(0)}{\lambda_1\cdots\lambda_l}\quad\text{as}\quad T\to+\infty. \tag{2.13}$$

From (2.8), (2.9), (2.12) and (2.13), one gets (2.7). □

We refer to [BV], [Bi2] and [BGV, Chap. 7] for the general case where the zero set of K may not be discrete.

2.2 Bott Residue Formula

We make the same assumptions as in the previous section. In particular, we still assume that the zero set of the Killing vector field K is discrete.

Let R^{TM} be the curvature of the Levi-Civita connection ∇^{TM}.

Let $i_1,\ldots,i_k$ be k positive even integers.

For any $p\in\mathrm{zero}(K)$ and $1\le j\le k$, set

$$\lambda^{i_j}(p)=\lambda_1^{i_j}+\cdots+\lambda_l^{i_j}.$$

One can state a version of the Bott residue formula [Bo], which reduces the computation of characteristic numbers of TM to quantities on zero(K), as follows.

Theorem 2.4 *If $i_1+\cdots+i_k=l$, then the following identity holds,*

$$\int_M \mathrm{tr}\left[\left(R^{TM}\right)^{i_1}\right]\cdots\mathrm{tr}\left[\left(R^{TM}\right)^{i_k}\right]$$

$$=\left(2\pi\sqrt{-1}\right)^l\sum_{p\in\mathrm{zero}(K)}\frac{2^k\lambda^{i_1}(p)\cdots\lambda^{i_k}(p)}{\lambda(p)}. \tag{2.14}$$

Moreover, if $i_1 + \cdots + i_k < l$, *then*

$$\sum_{p \in \mathrm{zero}(K)} \frac{\lambda^{i_1}(p) \cdots \lambda^{i_k}(p)}{\lambda(p)} = 0. \tag{2.15}$$

Proof. Clearly, the interior multiplication i_K can be extended canonically to an action on $\Omega^*(M; \mathrm{End}(TM))$. Also, since both ∇^{TM} and K are S^1-invariant, one sees directly that

$$\left[\nabla^{TM}, \mathcal{L}_K\right] = 0, \qquad \left[i_K, \mathcal{L}_K\right] = 0. \tag{2.16}$$

Moreover, one verifies directly that

$$\begin{aligned} \left(\nabla^{TM} + i_K\right)^2 &= R^{TM} + \left[\nabla^{TM}, i_K\right] \\ &= R^{TM} + \mathcal{L}_K + L_K, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} L_K &:= \nabla_K^{TM} - \mathcal{L}_K|_{\Gamma(TM)} = \nabla_K^{TM} - [K, \cdot] \\ &= \nabla_{\cdot}^{TM} K \in \Omega^0(M; \mathrm{End}(TM)). \end{aligned} \tag{2.18}$$

From (2.16) and (2.17), one gets the following Bianchi type formula

$$\left[\nabla^{TM} + i_K, R^{TM} + L_K\right] = 0. \tag{2.19}$$

From Lemma 1.8 and formulas (2.18), (2.19), one sees that for any integer h,

$$(d + i_K) \operatorname{tr}\left[\left(R^{TM} + L_K\right)^h\right] = \operatorname{tr}\left[\left[\nabla^{TM} + i_K, \left(R^{TM} + L_K\right)^h\right]\right] = 0.$$

This means that each $\operatorname{tr}[(R^{TM} + L_K)^{i_j}]$, $1 \le j \le k$, is d_K-closed. Their product is thus also d_K-closed.

One can then apply Theorem 2.3 to get

$$\begin{aligned} &\int_M \operatorname{tr}\left[\left(R^{TM}\right)^{i_1}\right] \cdots \operatorname{tr}\left[\left(R^{TM}\right)^{i_k}\right] \\ &= (2\pi)^l \sum_{p \in \mathrm{zero}(K)} \frac{\operatorname{tr}\left[\left(L_K(p)\right)^{i_1}\right] \cdots \operatorname{tr}\left[\left(L_K(p)\right)^{i_k}\right]}{\lambda(p)}. \end{aligned} \tag{2.20}$$

Now by (2.18) and the explicit expression of K given in the previous section, one sees that

$$(L_K(p))^2 = -\mathrm{diag}\left\{\lambda_1^2, \lambda_1^2, \cdots, \lambda_l^2, \lambda_l^2\right\}.$$

Thus, for each $1 \leq j \leq k$,

$$\mathrm{tr}\left[(L_K(p))^{i_j}\right] = 2(-1)^{i_j/2}\lambda^{i_j}(p). \tag{2.21}$$

Theorem 2.4 then follows from (2.20) and (2.21). □

The generalization of Theorem 2.4 in the case where the zero set of K may not be discrete was first proved by Baum and Cheeger in [BC].

2.3 Duistermaat-Heckman Formula

In this section, we further assume that M is a symplectic manifold with the symplectic form given by $\omega \in \Omega^2(M)$. We assume the S^1-action preserves ω. Moreover, we assume that the S^1-action on (M, ω) is *Hamiltonian.* That is, there exists a smooth function $\mu \in C^\infty(M)$ such that

$$d\mu = i_K\omega. \tag{2.22}$$

The **Liouville form** of the symplectic manifold (M, ω) is given by $\frac{\omega^l}{(2\pi)^l l!}$.

We still assume that the zero set of K is discrete.

The Duistermaat-Heckman formula [DH] can be stated as follows.

Theorem 2.5 *The following identity holds,*

$$\int_M \exp\left(\sqrt{-1}\mu\right)\frac{\omega^l}{(2\pi)^l l!} = \left(\sqrt{-1}\right)^l \sum_{p\in\mathrm{zero}(K)} \frac{\exp\left(\sqrt{-1}\mu(p)\right)}{\lambda(p)}. \tag{2.23}$$

Proof. From (2.22), one finds

$$(d + i_K)(\omega - \mu) = 0.$$

Thus, one sees that $\exp(\sqrt{-1}\mu - \sqrt{-1}\omega)$ is also d_K-closed. One can then

apply Theorem 2.3 to get

$$\int_M \exp\left(\sqrt{-1}\mu - \sqrt{-1}\omega\right) = (2\pi)^l \sum_{p\in \mathrm{zero}(K)} \frac{\exp\left(\sqrt{-1}\mu(p)\right)}{\lambda(p)},$$

from which (2.23) follows easily. □

The generalization of Theorem 2.5 in the case where the zero set of K may not be discrete is also due to Duistermaat-Heckman [DH].

It was proposed by Witten that a formal application of the Duistermaat-Heckman formula to the free loop space of a compact spin manifold can lead to a heuristic proof of the index theorem for the canonical Dirac operator on that spin manifold. Witten's idea was exposed in a talk of Atiyah [A] which in turn inspired Bismut [Bi1] to give a probabilistic proof of the index theorem for Dirac operators. The paper [Bi2] contains, among other things, the family generalizations of this circle of ideas.

2.4 Bott's Original Idea

Bott's original proof of Theorem 2.4 in [Bo] uses the idea of trangression, and is thus different from the one we presented above. Here we give a brief description of Bott's idea by re-proving Proposition 2.1.

Thus let ω be a d_K-closed form on M, with K has no zeros on M.

Let $\theta \in \Omega^1(M)$ be the one form on M such that for any $X \in \Gamma(TM)$, $i_X\theta = \langle X, K\rangle$.

Since

$$d_K\theta = |K|^2 + d\theta$$

and K is nowhere zero on M,

$$\frac{1}{d_K\theta} \in \Omega^*(M)$$

is well-defined.

From the fact that $d_K^2\theta = 0$, one then verifies directly that

$$\omega = d_K\left(\frac{\theta\wedge\omega}{d_K\theta}\right),$$

from which Proposition 2.1 follows directly from the Stokes formula.

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Chapter 3

Gauss-Bonnet-Chern Theorem

In this chapter we will present Mathai-Quillen's proof [MQ] of the Gauss-Bonnet-Chern theorem [C1], which expresses the Euler characteristic of a closed oriented Riemannian manifold as an integral of the Pfaffian of the curvature of the associated Levi-Civita connection. The proof is based on the explicit geometric constructions of Thom forms given in [MQ], while the basic idea behind is the same as in [C1]: *transgression.*

We will first construct the Thom form of Mathai-Quillen and then use it to prove the Gauss-Bonnet-Chern theorem.

We will work with real coefficients in this chapter.

3.1 A Toy Model and the Berezin Integral

We start with the simplest situation.

Let E be an oriented Euclidean vector space of dimension n, which we view as a vector bundle over a point. Let $\mathbf{x} = (x^1, \ldots, x^n)$ be an oriented Euclidean coordinate system of E. Set

$$U(\mathbf{x}) = e^{-|\mathbf{x}|^2/2} dx^1 \wedge \cdots \wedge dx^n. \tag{3.1}$$

Then one verifies easily that

$$\left(\frac{1}{2\pi}\right)^{n/2} \int_E U = 1. \tag{3.2}$$

We now reinterpret (3.2) in terms of the language of Berezin integral.

Let $\Lambda^*(E)$ be the exterior algebra of E.

The **Berezin integral** of an oriented Euclidean space E is a linear map

$$\int^B : \Lambda^*(E) \to \mathbf{R}$$

defined by

$$\int^B : \omega \in \Lambda^*(E) \mapsto \left\langle \omega, dx^1 \wedge \cdots \wedge dx^n \right\rangle, \tag{3.3}$$

which means that if $e_1, \cdots, e_n$ is an oriented orthonormal basis of E and $ae_1 \wedge \cdots \wedge e_n$ is the component of ω of degree n, then

$$\int^B \omega = a.$$

We lift $\Lambda^*(E)$ as a vector bundle over E, and denote by $\Omega^*(E, \Lambda^*(E))$ the space of smooth sections of $\Lambda^*(E)$ over E. Then we can and we will extend the Berezin integral to $\Omega^*(E, \Lambda^*(E))$ such that

$$\int^B : \alpha \wedge \beta \in \Omega^*\left(E, \Lambda^*(E)\right) \mapsto \alpha \int^B \beta \in \Omega^*(E), \tag{3.4}$$

with $\alpha \in \Omega^*(E)$, $\beta \in \Gamma(\Lambda^*(E))$.

We now consider the identity map $E \to E$ to be an element of $\Omega^0(E, E)$, with its exterior differential $d\mathbf{x} \in \Omega^1(E, E)$. The following result gives the Berezin integral interpretation of the differential form U defined in (3.1).

Proposition 3.1 *The following identity in $\Omega^*(E)$ holds,*

$$U(\mathbf{x}) = (-1)^{n(n+1)/2} \int^B \exp\left(-\frac{|\mathbf{x}|^2}{2} - d\mathbf{x}\right). \tag{3.5}$$

Proof. As $\mathbf{x} = x^1 e_1 + \cdots + x^n e_n$, one verifies directly that

$$(-1)^{n(n+1)/2} \int^B e^{-d\mathbf{x}} = (-1)^{n(n+1)/2} \int^B \prod_{k=1}^{n} \left(1 - dx^k \wedge e_k\right)$$

$$= (-1)^{n(n+1)/2} (-1)^n \int^B \left(dx^1 \wedge e_1\right) \wedge \cdots \wedge \left(dx^n \wedge e_n\right)$$

$$= dx^1 \wedge \cdots \wedge dx^n,$$

from which (3.5) follows. □

Finally, let E be an oriented Euclidean vector bundle of rank n over a manifold M. Then by an obvious fiberwise extension, we can extend the Berezin integral defined above to define a map

$$\int^B : \Omega^*(M, \Lambda^*(E)) \to \Omega^*(M) \tag{3.6}$$

in the way similar to that in (3.4). We still call it a Berezin integral.

Let ∇^E be a Euclidean connection on E (that is, ∇^E preserves the metric on E), then it extends naturally to an action ∇ on $\Omega^*(M, \Lambda^*(E))$.

The following property is important for the applications in the next section.

Proposition 3.2. *For any $\alpha \in \Omega^*(M, \Lambda^*(E))$, the following identity holds,*

$$d \int^B \alpha = \int^B \nabla \alpha. \tag{3.7}$$

Proof. Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of E. Without loss of generality, we can assume that $\alpha = \omega e_1 \wedge \cdots \wedge e_n$ with $\omega \in \Omega^*(M)$.

Now since ∇^E preserves the Euclidean metric on E, one verifies directly that

$$\nabla \alpha = (d\omega) e_1 \wedge \cdots \wedge e_n + (-1)^{\deg \omega} \omega \wedge \nabla(e_1 \wedge \cdots \wedge e_n)$$

$$= (d\omega) e_1 \wedge \cdots \wedge e_n,$$

from which (3.7) follows. □

3.2 Mathai-Quillen's Thom Form

In this section, we construct Mathai-Quillen's Thom form by using the Berezin integral.

Let M be an oriented closed manifold and $p : E \to M$ an oriented Euclidean vector bundle of rank n. Let ∇^E be a Euclidean connection on E. Then ∇^E lifts to a Euclidean connection on $p^* E$ and thus also to a derivation ∇ on $\Omega^*(E, \Lambda^*(p^* E))$.

On the other hand, for any $s \in \Gamma(E, p^*E)$, the interior multiplication i_s on $\Lambda^*(p^*E)$ extends naturally to a derivation on $\Omega^*(E, \Lambda^*(p^*E))$.

We will apply Proposition 3.2 to the triple (E, p^*E, ∇).

Since the interior multiplications decrease the degrees in $\Lambda^*(p^*E)$, from Proposition 3.2 one gets

$$d\int^B \alpha = \int^B (\nabla + i_s)\alpha, \tag{3.8}$$

for any $\alpha \in \Omega^*(E, \Lambda^*(p^*E))$ and $s \in \Gamma(E, p^*E)$.

Also, we identify $so(E)$, the subset of $\mathrm{End}(E)$ consisting of skew-adjoint elements, with $\Lambda^2(E)$ by the map

$$A \in so(E) \mapsto \sum_{i<j} \langle Ae_i, e_j\rangle e_i \wedge e_j. \tag{3.9}$$

We now consider the following elements in the algebra $\Omega^*(E, \Lambda^*(p^*E))$:

(1) the tautological section $\mathbf{x} \in \Omega^0(E, p^*E) = \Gamma(E, p^*E)$;
(2) the elements $|\mathbf{x}|^2 \in \Omega^0(E)$ and $\nabla\mathbf{x} \in \Omega^1(E, \Lambda^1(p^*E))$;
(3) the element $p^*R^E \in \Omega^2(E, \Lambda^2(p^*E))$ which is the pull back by p of the curvature $R^E = (\nabla^E)^2 \in \Omega^2(M, \Lambda^2(E))$, where we have used the identification (3.9).

The following result is of critical importance.

Lemma 3.3 *Let*

$$\mathcal{A} = \frac{|\mathbf{x}|^2}{2} + \nabla\mathbf{x} - p^*R^E \in \Omega^*(E, \Lambda^*(p^*E)).$$

Then

$$(\nabla + i_{\mathbf{x}})\,\mathcal{A} = 0. \tag{3.10}$$

Proof. By Leibniz's rule, we have

$$\nabla\left(|\mathbf{x}|^2\right) = -2i_{\mathbf{x}}\nabla\mathbf{x}.$$

By the definition of the curvature, we have

$$\nabla(\nabla\mathbf{x}) = \nabla^2\mathbf{x} = \left(p^*R^E\right)\mathbf{x} = i_{\mathbf{x}}p^*R^E,$$

while by the Bianchi identity we have

$$\nabla p^* R^E = 0.$$

Combining all these with the obvious fact that $i_{\mathbf{x}}|\mathbf{x}|^2 = 0$, we get (3.10). □

Following Mathai and Quillen [MQ], we now define a form U on E by

$$U = (-1)^{n(n+1)/2} \int^B e^{-\mathcal{A}}$$

$$= (-1)^{n(n+1)/2} \int^B e^{-\frac{|\mathbf{x}|^2}{2} - \nabla \mathbf{x} + p^* R^E}. \tag{3.11}$$

The following result shows that U is a **Thom form*** for E.

Proposition 3.4 *The form U is a closed n-form on E. Furthermore, one has the following formula for the fiberwise integration,*

$$\left(\frac{1}{2\pi}\right)^{n/2} \int_{E/M} U = 1. \tag{3.12}$$

Proof. Since

$$\mathcal{A} \in \bigoplus_{i=0}^{2} \Omega^i(E, \Lambda^i(p^* E)),$$

one gets

$$e^{-\mathcal{A}} \in \bigoplus_{i=0}^{n} \Omega^i \left(E, \Lambda^i \left(p^* E\right)\right).$$

By (3.8), (3.10) and (3.11), one verifies easily that U is a closed n-form on E.

To verify (3.12), one simply restricts to each fiber, on which one can apply directly (3.2) and (3.5). □

Mathai and Quillen [MQ] originally obtained their Thom form by computing the Chern character of Quillen's superconnection [Q] associated to

*We refer to the book of Bott-Tu [BoT] for the topological significance of the Thom forms and the associated classes.

spin vector bundles. The Berezin integral formalism here is adapted from [BGV, Sec. 1.6] and [BZ, Sec. 3].

3.3 A Transgression Formula

The Thom form U defined in the last section depends on the choices of the Euclidean metric and the connection ∇^E of E. However, one can show, by using certain transgression formula, that the cohomology class it determines (that is, the associated **Thom class**) does not depend on these metrics and connections.

For the proof of the Gauss-Bonnet-Chern theorem, here we only consider a special case of this transgression formula, i. e., the case where the metric on E being rescaled. This is equivalent to change $\mathbf{x}$ to $t\mathbf{x}$ in $\mathcal{A}$ for $t > 0$:

$$\mathcal{A}_t = \frac{t^2|\mathbf{x}|^2}{2} + t\nabla\mathbf{x} - p^*R^E. \tag{3.13}$$

Let U_t be the Thom form corresponding to $\mathcal{A}_t$ as being defined in (3.11)

Proposition 3.5 *We have the transgression formula*

$$\frac{dU_t}{dt} = -(-1)^{n(n+1)/2} d \int^B \left(\mathbf{x}e^{-\mathcal{A}_t}\right). \tag{3.14}$$

Proof. One verifies from (3.13) that

$$\frac{d\mathcal{A}_t}{dt} = t|\mathbf{x}|^2 + \nabla\mathbf{x} = (\nabla + ti_{\mathbf{x}})\,\mathbf{x}. \tag{3.15}$$

On the other hand, (3.10) now takes the form

$$(\nabla + ti_{\mathbf{x}})\mathcal{A}_t = 0. \tag{3.16}$$

From (3.15), (3.16), one deduces that

$$\frac{d}{dt}e^{-\mathcal{A}_t} = -\frac{d\mathcal{A}_t}{dt}e^{-\mathcal{A}_t} = -\left(\nabla + ti_{\mathbf{x}}\right)\left(\mathbf{x}e^{-\mathcal{A}_t}\right),$$

and hence, in view of (3.8),

$$\frac{dU_t}{dt} = -(-1)^{n(n+1)/2} \int^B \left(\nabla + ti_{\mathbf{x}}\right)\left(\mathbf{x}e^{-\mathcal{A}_t}\right)$$

$$= -(-1)^{n(n+1)/2} d \int^B \left(\mathbf{x} e^{-\mathcal{A}_t}\right). \tag{3.17}$$

□

3.4 Proof of the Gauss-Bonnet-Chern Theorem

We now assume that E is the tangent bundle TM of M and ∇^{TM} is the Levi-Civita connection associated to the metric g^{TM} on TM. Let R^{TM} be the curvature of ∇^{TM}.

We also assume that n, which now equals to the dimension of M, is an even integer.

Let $v \in \Gamma(TM)$ be a vector field on M. By (3.11) and Proposition 3.4, the pull-back v^*U is a closed differential form of degree n on M given by the formula

$$v^*U = (-1)^{n/2} \int^B \exp\left(-\left(\frac{|v|^2}{2} + \nabla^{TM} v - R^{TM}\right)\right). \tag{3.18}$$

In particular, if we take $v = 0$, the zero section, we get the so-called **Euler form**

$$(-1)^{n/2} \mathrm{Pf}\left(R^{TM}\right) := (-1)^{n/2} \int^B \exp\left(R^{TM}\right). \tag{3.19}$$

The following result shows that the cohomology class associated to $(-1)^{n/2}\mathrm{Pf}(R^{TM})$ does not depend on the choice of g^{TM}. We call this class the **Euler class** of TM.

Proposition 3.6 *If $\widetilde{g}^{TM}$ is another metric on TM, $\widetilde{\nabla}^{TM}$ is the Levi-Civita connection associated to $\widetilde{g}^{TM}$ and $\widetilde{R}^{TM}$ is the curvature of $\widetilde{\nabla}^{TM}$, then there exists a differential form $\omega \in \Omega^{n-1}(M)$ such that*

$$\mathrm{Pf}\left(R^{TM}\right) - \mathrm{Pf}\left(\widetilde{R}^{TM}\right) = d\omega. \tag{3.20}$$

Proof. For any $u \in [0,1]$, let g_u^{TM} be the metric on TM defined by

$$g_u^{TM} = u g^{TM} + (1-u)\widetilde{g}^{TM}.$$

Let ∇_u^{TM} be the Levi-Civita connection of g_u^{TM} and R_u^{TM} the curvature of ∇_u^{TM}.

By using Proposition 3.2, the Bianchi identity and (3.19), one deduces that

$$\frac{d}{du}\mathrm{Pf}\left(R_u^{TM}\right) = \int^B \frac{dR_u^{TM}}{du}\exp\left(R_u^{TM}\right)$$

$$= \int^B \left[\nabla_u^{TM}, \frac{d\nabla_u^{TM}}{du}\right]\exp\left(R_u^{TM}\right)$$

$$= \int^B \left[\nabla_u^{TM}, \frac{d\nabla_u^{TM}}{du}\exp\left(R_u^{TM}\right)\right]$$

$$= d\int^B \frac{d\nabla_u^{TM}}{du}\exp\left(R_u^{TM}\right),$$

from which one gets

$$\mathrm{Pf}\left(R^{TM}\right) - \mathrm{Pf}\left(\widetilde{R}^{TM}\right) = d\int_0^1\int^B \frac{d\nabla_u^{TM}}{du}\exp\left(R_u^{TM}\right)du,$$

which completes the proof of Proposition 3.6. □

Let $\chi(M)$ denote the Euler characteristic of M.

We can now state the Gauss-Bonnet-Chern theorem [C1] as follows.

Theorem 3.7 *The following identity holds,*

$$\chi(M) = \left(\frac{-1}{2\pi}\right)^{n/2}\int_M \mathrm{Pf}\left(R^{TM}\right). \tag{3.21}$$

Proof. Let V be a transversal section of TM. That is, V is a tangent vector field on M such that the zero set of V, denoted by $\mathrm{zero}(V)$, is discrete and nondegenerate. The later means that for any $p \in \mathrm{zero}(V)$, there is an oriented coordinate system $\mathbf{y} = (y^1, \ldots, y^n)$ on a sufficiently small open neighborhood U_p of p such that near p,

$$V(\mathbf{y}) = \mathbf{y}A + O\left(|\mathbf{y}|^2\right), \tag{3.22}$$

where A is an $n \times n$ matrix not depending on $\mathbf{y}$ verifying that

$$\det(A) \neq 0. \tag{3.23}$$

The existence of such a transversal section is an elementary result in differential topology.

In order to prove (3.21), observe that by (3.18), (3.19) and Proposition 3.5, one has that for any $t > 0$,

$$(-1)^{n/2} \int_M \mathrm{Pf}\left(R^{TM}\right)$$

$$= (-1)^{n/2} \int_M \int^B \exp\left(-\left(\frac{t^2|V|^2}{2} + t\nabla^{TM} V - R^{TM}\right)\right). \tag{3.24}$$

For any $p \in \mathrm{zero}(V)$, from (3.22) one sees easily that one can modify the coordinate system slightly so that (3.22) becomes

$$V(\mathbf{y}) = \mathbf{y}A. \tag{3.25}$$

Moreover, by Proposition 3.6, we can well assume that on U_p the metric g^{TM} is of the form

$$g^{TM} = \left(dy^1\right)^2 + \cdots + \left(dy^n\right)^2.$$

With these simplifying assumptions we can rewrite (3.24) as

$$(-1)^{n/2} \int_M \mathrm{Pf}\left(R^{TM}\right)$$

$$= (-1)^{n/2} \sum_{p\in \mathrm{zero}(V)} \int_{U_p} \int^B \exp\left(-\left(\frac{t^2|V|^2}{2} + tdV\right)\right)$$

$$+(-1)^{n/2} \int_{M\setminus \cup_{p\in\mathrm{zero}(V)} U_p} \int^B \exp\left(-\left(\frac{t^2|V|^2}{2} + t\nabla^{TM} V - R^{TM}\right)\right). \tag{3.26}$$

Since $|V| > 0$ has a positive lower bound on $M \setminus \cup_{p\in\mathrm{zero}(V)} U_p$, one sees easily that as $t \to +\infty$, one has

$$\int_{M\setminus \cup_{p\in\mathrm{zero}(V)} U_p} \int^B \exp\left(-\left(\frac{t^2|V|^2}{2} + t\nabla^{TM} V - R^{TM}\right)\right) \longrightarrow 0. \tag{3.27}$$

On the other hand, for any zero p of V, one verifies directly that, as $t \to +\infty$,

$$(-1)^{n/2} \int_{U_p} \int^B \exp\left(- \left(\frac{t^2 |V|^2}{2} + tdV \right) \right)$$

$$= (-1)^{n/2} \int_{U_p} \int^B \exp\left(- \left(\frac{t^2 |\mathbf{y}A|^2}{2} + td(\mathbf{y}A) \right) \right)$$

$$= t^n \det(A) \int_{U_p} \exp\left(- \left(\frac{t^2 |\mathbf{y}A|^2}{2} \right) \right) dy^1 \wedge \cdots \wedge dy^n$$

$$\longrightarrow \operatorname{sgn}(\det(A)) \int_{\mathbf{R}^n} \exp\left(- \left(\frac{|\mathbf{y}A|^2}{2} \right) \right) |\det(A)| dy^1 \wedge \cdots \wedge dy^n$$

$$= (2\pi)^{n/2} \operatorname{sgn}(\det(A)). \tag{3.28}$$

Now recall the classical Poincaré-Hopf index formula (cf. [BoT, Theorem 11.25]) which says that

$$\chi(M) = \sum_{p \in \operatorname{zero}(V)} \operatorname{sgn}(\det(A_p)). \tag{3.29}$$

By (3.24) and (3.26)-(3.29), one gets (3.21). □

3.5 Some Remarks

Remark 3.8 To see more closely the relationship between the above proof and Chern's original proof in [C1], one integrates both sides of the transgression formula (3.14) to get for $E = TM$ and any $T > 0$ that

$$\left(\frac{-1}{2\pi} \right)^{n/2} \int^B \exp\left(p^* R^{TM} \right) - \left(\frac{-1}{2\pi} \right)^{n/2} \int^B \exp(-\mathcal{A}_T)$$

$$= \left(\frac{-1}{2\pi} \right)^{n/2} d \int_0^T \int^B \left(\mathbf{x} e^{-\mathcal{A}_t} \right) dt. \tag{3.30}$$

Now if one restricts (3.30) to the unit sphere bundle SM of TM, one verifies directly that when $T \to +\infty$,

$$\left(\frac{-1}{2\pi}\right)^{n/2} \int^B \exp(-\mathcal{A}_T) \to 0.$$

Thus, when restricted to the unit sphere bundle SM, one has

$$\left(\frac{-1}{2\pi}\right)^{n/2} p^*\mathrm{Pf}\left(R^{TM}\right) = \left(\frac{-1}{2\pi}\right)^{n/2} d\int_0^{+\infty} \int^B \left(\mathbf{x}e^{-\mathcal{A}_t}\right) dt, \tag{3.31}$$

which looks of exactly the same form as Chern's transgression formula in [C1]. We leave it to the interested reader to show that they are in fact the same one.

Remark 3.9 There is also a heat kernel proof of the Gauss-Bonnet-Chern theorem due to Patodi [P], see the books [BGV] and [Y] for more details. On the other hand, there is an analytic proof of the Poincaré-Hopf index formula (3.29) suggested by Witten [W]. We will present such a proof in the next chapter.

3.6 Chern's Original Proof

In this section we describe Chern's simple and elegant original proof of (3.31), from which Theorem 3.7 follows from the Poincaré-Hopf index formula and also the Stokes formula. Here, instead of using the arguments in [C1], we adopt a simplified version due to Chern himself [C2].

We make the same assumptions and use the same notation as in previous sections.

Recall that SM is the unit sphere bundle of the tangent bundle $p : TM \to M$. Thus, $\mathbf{x}$ forms a unit length section over SM of p^*TM. We denote this section by e_n. Let $e_1, \ldots, e_{n-1}$ be the (locally defined) sections of p^*TM over SM so that $e_1, \ldots, e_{n-1}, e_n$ forms an oriented orthonormal basis of p^*TM over SM.

For any integer i, j such that $1 \leq i, j \leq n$, let ω_{ij} be the (locally defined) one form on SM define by

$$\nabla e_i = \sum_{j=1}^{n} \omega_{ij} e_j. \tag{3.32}$$

Let

$$\Omega_{ij} = \langle p^* R^{TM} e_i, e_j \rangle$$

denote the (locally defined) two form on SM.

From (3.32), one verifies easily that

$$\Omega_{ij} = d\omega_{ij} - \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj}, \tag{3.33}$$

from which one finds the following (local) version of the Bianchi identity,

$$d\Omega_{ij} = -\sum_{k=1}^{n} \Omega_{ik} \wedge \omega_{kj} + \sum_{k=1}^{n} \omega_{ik} \wedge \Omega_{kj}. \tag{3.34}$$

For any l integers $\alpha_1, \ldots, \alpha_l$, let $\epsilon_{\alpha_1 \ldots \alpha_l}$ be 1 or -1 if $\alpha_1, \ldots, \alpha_l$ is an even or odd permutation of $1, \ldots, l$ respectively. Otherwise we define it to be zero.

Following Chern [C2, (4)], for any integer k such that $0 \leq k \leq \frac{n}{2} - 1$, we define

$$\Phi_k = \sum_{\alpha_1, \cdots, \alpha_{n-1}=1}^{n-1} \epsilon_{\alpha_1 \ldots \alpha_{n-1}} \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}\alpha_{2k}} \wedge \omega_{\alpha_{2k+1} n} \wedge \cdots \wedge \omega_{\alpha_{n-1} n},$$

$$\Psi_k = 2(k+1) \sum_{\alpha_1, \cdots, \alpha_{n-1}=1}^{n-1} \epsilon_{\alpha_1 \ldots \alpha_{n-1}} \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}\alpha_{2k}}$$

$$\wedge \Omega_{\alpha_{2k+1} n} \wedge \omega_{\alpha_{2k+2} n} \wedge \cdots \wedge \omega_{\alpha_{n-1} n}. \tag{3.35}$$

We also define $\Psi_{-1} = \Psi_{n/2} = 0$.

One verifies directly that each of the Φ_k's and Ψ_k's does not depend on the choice of $e_1, \ldots, e_{n-1}$, and thus is a globally well-defined differential form on SM. Thus, $d\Phi_k$ is also a globally well-defined differential form on SM, which, by (3.35), is given by

$$d\Phi_k = k \sum_{\alpha_1, \cdots, \alpha_{n-1}=1}^{n-1} \epsilon_{\alpha_1 \ldots \alpha_{n-1}} d\Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}\alpha_{2k}} \wedge \omega_{\alpha_{2k+1} n} \wedge \cdots \wedge \omega_{\alpha_{n-1} n}$$

$$+(n-2k-1)\sum_{\alpha_1,\cdots,\alpha_{n-1}=1}^{n-1}\epsilon_{\alpha_1\ldots\alpha_{n-1}}\Omega_{\alpha_1\alpha_2}\wedge\cdots\wedge\Omega_{\alpha_{2k-1}\alpha_{2k}}$$

$$\wedge d\omega_{\alpha_{2k+1}n}\wedge\cdots\wedge\omega_{\alpha_{n-1}n}. \tag{3.36}$$

By (3.32), one finds easily that

$$\omega_{nn}=0.$$

Now we substitute $d\Omega_{\alpha_1\alpha_2}$ and $d\omega_{\alpha_{2k+1}n}$ in (3.36) by the right hand sides of (3.33) and (3.34) respectively. Since $d\Phi_k$ is a global form on SM, we see that the terms involving (locally defined) $\omega_{\alpha\beta}$, with $1\le\alpha,\ \beta\le n-1$, should cancel each other.†

In summary, from (3.36) one deduces that

$$d\Phi_k=\Psi_{k-1}+\frac{n-2k-1}{2(k+1)}\Psi_k. \tag{3.37}$$

Following [C2, (9) and (10)], we define

$$\Pi=\left(\frac{1}{2\pi}\right)^{n/2}\sum_{k=1}^{\frac{n}{2}-1}\frac{(-1)^k}{1\cdot3\cdots(n-2k-1)\cdot2^k\cdot k!}\Phi_k \tag{3.38}$$

and

$$\Omega=\left(\frac{-1}{2\pi}\right)^{n/2}\frac{1}{2^{n/2}\left(\frac{n}{2}\right)!}\sum_{i_1,\cdots,i_n=1}^{n}\epsilon_{i_1\ldots i_n}\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{n-1}i_n}. \tag{3.39}$$

By (3.37), one then gets as in [C2, (11)] the following transgression formula on SM,

$$-d\Pi=\Omega, \tag{3.40}$$

which is exactly (3.31). □

Remark 3.10 The historical importance of Chern's proof, besides introducing the idea of transgression, is that it was the first time in history that the intrinsically defined sphere bundle was used to solve an important problem in geometry.

†In fact, for any $p\in SM$, one can find $e_1,\ldots,e_{n-1}$ near p so that $\omega_{\alpha\beta}(p)=0$ for $1\le\alpha,\ \beta\le n-1$ (cf. [CCL, Theorem 4.1.2]).

Remark 3.11 Although the Mathai-Quillen formalism provides a reasonable interpretation of Chern's transgression formula through the geometric construction of Thom forms, it is still mysterious how Chern constructed the forms Φ_k's and Ψ_k's, especially under the form appeared in [C1].

Remark 3.12 There is also a generalization of the Gauss-Bonnet-Chern theorem to Finsler manifolds, see the paper of Bao and Chern [BC].

3.7 References

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Chapter 4

Poincaré-Hopf Index Formula: an Analytic Proof

We have seen in the previous chapter that the Poincaré-Hopf index formula (3.29) plays an important role in the proof of the Gauss-Bonnet-Chern theorem. In this chapter, following an idea of Edward Witten [Wi], we will present a purely analytic proof of this classical result.

The strategy of Witten's proof is very simple. One starts with the analytic interpretation of the Euler characteristic obtained from the Hodge theorem, and deforms the involved elliptic operators by the vector field in question. In this process, one finds that the proof can be localized to sufficiently small neighborhoods of the zero set of the vector field. A further investigation on these small neighborhoods will then complete the proof.

In this chapter, we will first review the Hodge theorem and the consequent analytic interpretation of the Euler characteristic. We then introduce Witten's deformation and show how it leads to a proof of the Poincaré-Hopf index formula.

We work with real coefficients in this and next chapters.

4.1 Review of Hodge Theorem

Let M be an n-dimensional closed oriented manifold. Recall that the de Rham cohomology of M has been defined in Section 1.1.* The theorem of Hodge provides an analytic realization of this cohomology group.

To begin with, let g^{TM} be a metric on TM. Then the **Hodge star**

*Although in Section 1.1 we worked with complex coefficients. The same strategy works for real coefficients parallelly.

operator

$$* : \Lambda^*(T^*M) \to \Lambda^{n-*}(T^*M)$$

can be defined as follows: if $e_1, \ldots, e_n$ is an oriented orthonormal basis of TM and $e^1, \ldots, e^n$ is the corresponding dual basis in T^*M determined by g^{TM}, then for any integer k between 1 and n,

$$* : e^1 \wedge \cdots \wedge e^k \mapsto e^{k+1} \wedge \cdots \wedge e^n. \tag{4.1}$$

It can be verified easily that the operator $*$ is well-defined.

The following properties of the Hodge star operator are easy to verify.

(i) For any integer k, $** = (-1)^{nk+k} : \Omega^k(M) \to \Omega^k(M)$;
(ii) For any integer k and any α, $\beta \in \Omega^k(M)$, $\alpha \wedge *\beta = \beta \wedge *\alpha$;
(iii) $\alpha \wedge *\alpha = 0$ if and only if $\alpha = 0$.

From these properties, one can define an inner product $\langle \cdot, \cdot \rangle$ on $\Omega^*(M)$ as follows: for any α, $\beta \in \Omega^*(M)$,

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta. \tag{4.2}$$

Recall that $d : \Omega^*(M) \to \Omega^*(M)$ is the exterior differential operator on M.

Definition 4.1 Let $d^* : \Omega^*(M) \to \Omega^*(M)$ be the operator defined by

$$\alpha \in \Omega^k(M) \mapsto (-1)^{nk+n+1} * d * \alpha \in \Omega^{k-1}(M). \tag{4.3}$$

From (4.3) and Property (i) above, one verifies that for any $\alpha \in \Omega^k(M)$, $\beta \in \Omega^{k+1}(M)$,

$$d(\alpha \wedge *\beta) = (d\alpha) \wedge *\beta + (-1)^k \alpha \wedge d * \beta$$

$$= (d\alpha) \wedge *\beta - \alpha \wedge *d^*\beta,$$

from which one gets

$$\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle. \tag{4.4}$$

That is, d^* is the *formal adjoint* of d.

Definition 4.2 The **de Rham-Hodge operator** D associated to g^{TM} is defined by

$$D = d + d^* : \Omega^*(M) \longrightarrow \Omega^*(M).$$

Set

$$\Omega^{\mathrm{even}}(M) = \bigoplus_{i \text{ even}} \Omega^i(M), \qquad \Omega^{\mathrm{odd}}(M) = \bigoplus_{i \text{ odd}} \Omega^i(M).$$

Let

$$D_{\mathrm{even/odd}} : \Omega^{\mathrm{even/odd}}(M) \to \Omega^{\mathrm{odd/even}}(M)$$

be the restrictions of D to $\Omega^{\mathrm{even/odd}}(M)$ respectively. Clearly, D_{odd} is the formal adjoint of D_{even}.

Let

$$\Box = D^2 = dd^* + d^*d \tag{4.5}$$

be the **Laplacian** of D. Then $\Box$ preserves each $\Omega^k(M)$, $0 \leq k \leq n$.

We can now state the **Hodge decomposition theorem** as follows.

Theorem 4.3 *The following decomposition formula for $\Omega^*(M)$ holds,*

$$\Omega^*(M) = \ker \Box \oplus \operatorname{Im} \Box.$$

We refer to the books of de Rham [de] and Warner [W] for a proof of this result.

When restricted to each $\Omega^k(M)$ with $0 \leq k \leq n$, one further has

$$\Omega^k(M) = \ker \Box_k \oplus \operatorname{Im} d_{k-1} \oplus \operatorname{Im} d_{k+1}^*, \tag{4.6}$$

where we use the notation that for any integrer i such that $0 \leq i \leq n$, $\Box_i = \Box|_{\Omega^i(M)}$, $d_i = d|_{\Omega^i(M)}$ and $d_i^* = d^*|_{\Omega^i(M)}$.

Corollary 4.4 *For any integer k such that $0 \leq k \leq n$, one has the identification*

$$\ker \Box_k \simeq H^k_{\mathrm{dR}}(M; \mathbf{R}).$$

Proof. First, if $\omega \in \ker \Box_k$, then by (4.4) and (4.5), one has

$$\langle d\omega, d\omega\rangle + \langle d^*\omega, d^*\omega\rangle = \langle (d^*d + dd^*)\,\omega, \omega\rangle = 0, \tag{4.7}$$

which implies that $d\omega = 0$. Furthermore, if ω, $\omega' \in \ker \Box_k$ such that $\omega - \omega' = d\omega''$ for some $\omega'' \in \Omega^{k-1}$, then by (4.6) one sees that $\omega = \omega'$. That is, each element in $\ker \Box_k$ determines a unique element in $H^k_{\mathrm{dR}}(M;\mathbf{R})$.

On the other hand, if $d\omega = 0$, then by (4.6), one deduces easily that there is the unique decomposition $\omega = \omega' + d\omega''$ with $\omega' \in \ker \Box_k$ and $\omega'' \in \Omega^{k-1}(M)$, which determines an element in $\ker \Box_k$.

Combining the above discussions one completes the proof of Corollary 4.4. □

Now by (4.5)-(4.7) one also deduces that

$$\ker \Box = \ker\,(d + d^*) \subset \Omega^*(M). \tag{4.8}$$

From (4.8) and Corollary 4.4 one gets the following analytic interpretation of the Euler characteristic $\chi(M)$ of M:

$$\chi(M) := \sum_{i=0}^{n} (-1)^i \dim H^i_{\mathrm{dR}}(M;\mathbf{R})$$

$$= \mathrm{ind}\,\left(D_{\mathrm{even}} = d + d^* : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M)\right), \tag{4.9}$$

which by definition equals to

$$\dim\left(\ker D_{\mathrm{even}}\right) - \dim\left(\ker D_{\mathrm{odd}}\right).$$

4.2 Poincaré-Hopf Index Formula

For convenience we here restate the Poincaré-Hopf index formula.

Let V be a transversal section of TM. Then the zero set of V, denoted by zero(V), is discrete and for any $p \in \mathrm{zero}(V)$, there is a sufficiently small neighborhood U_p of p and an oriented coordinate system $\mathbf{y} = (y^1, \ldots, y^n)$ such that on U_p,

$$V(\mathbf{y}) = \mathbf{y} A_p \tag{4.10}$$

for some constant matrix A_p such that

$$\det A_p \neq 0,$$

and that the U_p's with $p \in \mathrm{zero}(V)$ are disjoint with each other.

The Poincaré-Hopf index formula (cf. [BoT, Theorem 11.25]) can be stated as follows.

Theorem 4.5 *The following identity holds,*

$$\chi(M) = \sum_{p \in \mathrm{zero}(V)} \mathrm{sgn}(\det(A_p)). \tag{4.11}$$

4.3 Clifford Actions and the Witten Deformation

For any $e \in TM$, let $e^* \in T^*M$ corresponds to e via g^{TM}. That is, for any $X \in \Gamma(TM)$, $(e^*, X) = \langle e, X\rangle$.

Let $c(e)$, $\widehat{c}(e)$ be the Clifford operators acting on the exterior algebra bundle $\Lambda^*(T^*M)$ defined by

$$c(e) = e^* \wedge -i_e, \quad \widehat{c}(e) = e^* \wedge +i_e, \tag{4.12}$$

where $e^*\wedge$ and i_e are the standard notation for exterior and interior multiplications.

If e, $e' \in TM$, one has

$$c(e)c(e') + c(e')c(e) = -2\langle e, e'\rangle,$$

$$\widehat{c}(e)\widehat{c}(e') + \widehat{c}(e')\widehat{c}(e) = 2\langle e, e'\rangle,$$

$$c(e)\widehat{c}(e') + \widehat{c}(e')c(e) = 0. \tag{4.13}$$

Let ∇^{TM} be the Levi-Civita connection associated to the metric g^{TM}. Then it induces canonically a Euclidean connection $\nabla^{\Lambda^*(T^*M)}$ on $\Lambda^*(T^*M)$.

Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of TM. Let $e^1, \ldots, e^n$ be the corresponding dual basis of T^*M with respect to g^{TM}.

Since ∇^{TM} is torsion free, one verifies directly that

$$d = \sum_{i=1}^{n} e^i \wedge \nabla_{e_i}^{\Lambda^*(T^*M)} : \Omega^*(M) \longrightarrow \Omega^*(M). \tag{4.14}$$

From (4.14), one deduces that for any $\alpha,\ \beta \in \Omega^*(M)$,

$$\int_M \left(d\alpha \wedge *\beta + \alpha \wedge * \sum_{i=1}^n i_{e_i} \nabla_{e_i}^{\Lambda^*(T^*M)} \beta \right)$$

$$= \int_M \sum_{i=1}^n \left(e^i \wedge \nabla_{e_i}^{\Lambda^*(T^*M)} \alpha \wedge *\beta + e^i \wedge \alpha \wedge * \nabla_{e_i}^{\Lambda^*(T^*M)} \beta \right)$$

$$= \int_M \sum_{i=1}^n e^i \wedge \nabla_{e_i}^{\Lambda^*(T^*M)} (\alpha \wedge *\beta)$$

$$= \int_M d(\alpha \wedge *\beta) = 0.$$

Thus, $-\sum_{i=1}^n i_{e_i} \nabla_{e_i}^{\Lambda^*(T^*M)}$ is a formal adjoint of d.

Since we have seen that d^* is a formal adjoint of d, by the uniqueness of the formal adjoint operators, one gets

$$d^* = -\sum_{i=1}^n i_{e_i} \nabla_{e_i}^{\Lambda^*(T^*M)} : \Omega^*(M) \longrightarrow \Omega^*(M). \tag{4.15}$$

From (4.12), (4.14) and (4.15), one gets

$$d + d^* = \sum_{i=1}^n c(e_i) \nabla_{e_i}^{\Lambda^*(T^*M)} : \Omega^*(M) \to \Omega^*(M). \tag{4.16}$$

Now let $V \in \Gamma(TM)$. Following Witten [Wi], for any $T \in \mathbf{R}$, set

$$D_T = d + d^* + T\widehat{c}(V) : \Omega^*(M) \to \Omega^*(M). \tag{4.17}$$

Then D_T is a (formally) self-adjoint first order elliptic differential operator. Clearly, D_T exchanges $\Omega^{\mathrm{even}}(M)$ and $\Omega^{\mathrm{odd}}(M)$. Let $D_{T,\mathrm{even/odd}}$ be the restrictions of D_T on $\Omega^{\mathrm{even/odd}}(M)$ respectively. Then $D_{T,\mathrm{odd}}$ is the formal adjoint of $D_{T,\mathrm{even}}$.

By a standard fact for elliptic operators, and by (4.9), one has that for any $T \in \mathbf{R}$,

$$\mathrm{ind}\, D_{T,\mathrm{even}} = \mathrm{ind}\, D_{\mathrm{even}} = \chi(M). \tag{4.18}$$

The following Bochner type formula for D_T^2 is crucial.

Proposition 4.6 *For any $T \in \mathbf{R}$, the following identity holds,*

$$D_T^2 = D^2 + T\sum_{i=1}^{n} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} V\right) + T^2|V|^2. \tag{4.19}$$

Proof. From (4.13), (4.16) and (4.17), one deduces that

$$D_T^2 = D^2 + T\sum_{i=1}^{n}\left(c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)}\widehat{c}(V) + \widehat{c}(V)c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)}\right) + T^2|V|^2$$

$$= D^2 + T\sum_{i=1}^{n}\left(c(e_i)\widehat{c}(V)\nabla_{e_i}^{\Lambda^*(T^*M)} + c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM}V\right) + \widehat{c}(V)c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)}\right)$$

$$+T^2|V|^2$$

$$= D^2 + T\sum_{i=1}^{n} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM}V\right) + T^2|V|^2.$$

□

4.4 An Estimate Outside of $\cup_{p\in \mathrm{zero}(V)}U_p$

Let $\|\cdot\|_0$ denote the 0-th Sobolev norm on $\Omega^*(M)$ induced by the inner product (4.2). Denote by $\mathbf{H}^0(M)$ the corresponding Sobolev space.

Let $V \in \Gamma(TM)$ be as in Section 4.2.

The main result of this section can be stated as follows.

Proposition 4.7 *There exist constants $C > 0$, $T_0 > 0$ such that for any section $s \in \Omega^*(M)$ with* $\mathrm{Supp}(s) \subset M \setminus \cup_{p\in \mathrm{zero}(V)}U_p$ *and $T \geq T_0$, one has*

$$\|D_T s\|_0 \geq C\sqrt{T}\|s\|_0. \tag{4.20}$$

Proof. Since V is nowhere zero on $M \setminus \cup_{p\in \mathrm{zero}(V)}U_p$, there is a constant $C_1 > 0$ such that on $M \setminus \cup_{p\in \mathrm{zero}(V)}U_p$,

$$|V|^2 \geq C_1. \tag{4.21}$$

From (4.19) and (4.21), one sees that there exists a constant $C_2 > 0$ such that

$$\|D_T s\|_0^2 = \langle D_T^2 s, s\rangle \geq \left(C_1 T^2 - C_2 T\right) \|s\|_0^2 \tag{4.22}$$

for any $s \in \Omega^*(M)$ with $\mathrm{Supp}(s) \subset M \setminus \cup_{p\in \mathrm{zero}(V)} U_p$.

Formula (4.20) then follows easily. □

4.5 Harmonic Oscillators on Euclidean Spaces

Proposition 4.7 indicates that the proof of the Poincaré-Hopf index formula can be 'localized' in some sense to small neighborhoods of zero(V). Without loss of generality, we assume that on each U_p, the metric g^{TM} is of the form

$$g^{TM} = \left(dy^1\right)^2 + \cdots + \left(dy^n\right)^2.$$

Then each U_p may be viewed as an open neighborhood of the n-dimensional Euclidean space E_n. In this section we investigate the Witten deformation in this Eulidean space for the vector field $V = \mathbf{y}A$ with $\det A \neq 0$.

Let $e_i = \frac{\partial}{\partial y^i}$, $1 \leq i \leq n$, be an oriented orthonormal basis of E_n.

Equation (4.19) can be written explicitly here as

$$\begin{aligned} D_T^2 &= -\sum_{i=1}^n \left(\frac{\partial}{\partial y^i}\right)^2 + T\sum_{i=1}^n c(e_i)\widehat{c}(e_iA) + T^2 \langle \mathbf{y}AA^*, \mathbf{y}\rangle \\ &= -\sum_{i=1}^n \left(\frac{\partial}{\partial y^i}\right)^2 - T\mathrm{Tr}\left[\sqrt{AA^*}\right] + T^2 \langle \mathbf{y}AA^*, \mathbf{y}\rangle \\ &\quad + T\left(\mathrm{Tr}\left[\sqrt{AA^*}\right] + \sum_{i=1}^n c(e_i)\widehat{c}(e_iA)\right). \end{aligned} \tag{4.23}$$

The operator

$$K_T = -\sum_{i=1}^n \left(\frac{\partial}{\partial y^i}\right)^2 - T\mathrm{Tr}\left[\sqrt{AA^*}\right] + T^2 \langle \mathbf{y}AA^*, \mathbf{y}\rangle \tag{4.24}$$

is a rescaled harmonic oscillator. By standard results concerning harmonic oscillators (cf. [GJ, Theorem 1.5.1]), one knows that when $T > 0$, K_T is a

nonnegative elliptic operator with $\ker K_T$ being of dimension one and being generated by

$$\exp\left(\frac{-T|\mathbf{y}A|^2}{2}\right). \tag{4.25}$$

Furthermore, the nonzero eigenvalues of K_T are all greater than CT for some fixed constant $C > 0$.

On the other hand, one has the following algebraic result due to S. P. Novikov (cf. [S]).

Lemma 4.8 *The linear operator*

$$L = \mathrm{Tr}\left[\sqrt{AA^*}\right] + \sum_{i=1}^{n} c(e_i)\widehat{c}(e_i A) \tag{4.26}$$

acting on $\Lambda^*(\mathrm{E}_n^*)$ *is nonnegative. Moreover,* $\dim(\ker L) = 1$ *with* $\ker L \subset \Lambda^{\mathrm{even}}(\mathrm{E}_n^*)$ *if* $\det A > 0$, *while* $\ker L \subset \Lambda^{\mathrm{odd}}(\mathrm{E}_n^*)$ *if* $\det A < 0$.

Proof. We write

$$A = U\sqrt{A^*A}$$

with $U \in O(n)$. Also, let $W \in SO(n)$ be such that

$$\sqrt{A^*A} = W\mathrm{diag}\{s_1, \ldots, s_n\}W^*,$$

where $\mathrm{diag}\{s_1, \ldots, s_n\}$ denotes the diagonal matrix with each $s_i > 0$, $1 \le i \le n$. Then one deduces that

$$\mathrm{Tr}\left[\sqrt{AA^*}\right] = \sum_{i=1}^{n} s_i \tag{4.27}$$

and

$$\sum_{i=1}^{n} c(e_i)\widehat{c}(e_i A) = \sum_{i=1}^{n} c(e_i)\widehat{c}\left(e_i UW\mathrm{diag}\{s_1, \ldots, s_n\}W^*\right). \tag{4.28}$$

Now write

$$UW = \{w_{ij}\}_{n\times n}.$$

From (4.28), one gets

$$\sum_{i=1}^{n} c(e_i)\widehat{c}(e_iA) = \sum_{i,j=1}^{n} c(e_i)\widehat{c}\left(e_j w_{ij} s_j W^*\right)$$

$$= \sum_{j=1}^{n} s_j c\left(e_j W^* U^*\right) \widehat{c}\left(e_j W^*\right). \tag{4.29}$$

Set $f_j = e_j W^*$, $1 \le j \le n$. They form another oriented orthonormal basis of E_n. From (4.26), (4.27) and (4.29), one finds

$$L = \sum_{i=1}^{n} s_i \left(1 + c\left(f_i U^*\right) \widehat{c}(f_i)\right). \tag{4.30}$$

Now for any integer j such that $1 \le j \le n$, set

$$\eta_j = c\left(f_j U^*\right) \widehat{c}(f_j).$$

Then by (4.13) one verifies easily that each η_j, $1 \le j \le n$, is self-adjoint and that $\eta_j^2 = 1$. Thus the lowest eigenvalue of each η_j, $1 \le j \le n$, is -1. This proves that the operator L in (4.30) is a nonnegative operator.

On the other hand, by (4.13) one also verifies $\eta_i \eta_j = \eta_j \eta_i$ for $1 \le i, j \le n$. Moreover, one verifies that $\widehat{c}(f_j)\eta_j = -\eta_j \widehat{c}(f_j)$, while $\widehat{c}(f_j)\eta_i = \eta_i \widehat{c}(f_j)$ when $i \ne j$.

From these two facts one deduces easily, via induction, that

$$\dim\left\{x \in \Lambda^*\left(\mathrm{E}_n^*\right) : (1 + \eta_j)x = 0 \quad \text{for} \quad 1 \le j \le n\right\} = \frac{\dim \Lambda^*\left(\mathrm{E}_n^*\right)}{2^n} = 1.$$

Moreover, let $\rho \in \Lambda^*(\mathrm{E}_n^*)$ denote one of the unit sections of $\ker L$, then one has

$$\rho = (-1)^n \left(\prod_{i=1}^{n} \eta_i\right) \rho$$

$$= (-1)^n (\det U) \left(\prod_{i=1}^{n} c(f_i)\widehat{c}(f_i)\right) \rho. \tag{4.31}$$

Now it is easy to see that

$$(-1)^n \left(\prod_{i=1}^{n} c(f_i)\widehat{c}(f_i) \right) \Bigg|_{\Lambda^{\mathrm{even/odd}}(\mathrm{E}_n^*)} = \pm \mathrm{Id}|_{\Lambda^{\mathrm{even/odd}}(\mathrm{E}_n^*)}. \tag{4.32}$$

From (4.31) and (4.32), one sees that $\rho \in \Lambda^{\mathrm{even/odd}}(\mathrm{E}_n^*)$ if and only if $\det(U) = \pm 1$.

This completes the proof of Lemma 4.8. □

Combining Lemma 4.8 with the properties of the (rescaled) harmonic oscillator K_T, one gets (Compare with [S, Corollary 2.22])

Proposition 4.9 *For any $T > 0$, the operator*

$$-\sum_{i=1}^{n} \left(\frac{\partial}{\partial y^i} \right)^2 + T \sum_{i=1}^{n} c(e_i)\widehat{c}(e_i A) + T^2 \langle \mathbf{y} A A^*, \mathbf{y} \rangle$$

acting on $\Gamma(\Lambda^*(\mathrm{E}_n^*))$ is nonnegative. Its kernel is of dimension one and is generated by

$$\exp\left(\frac{-T|\mathbf{y}A|^2}{2} \right) \cdot \rho. \tag{4.33}$$

Moreover, all the nonzero eigenvalues of this operator are greater than CT for some fixed constant $C > 0$.

4.6 A Proof of the Poincaré-Hopf Index Formula

We will use a simplified version of the analytic techniques developed by Bismut and Lebeau (cf. [BL, Chap. 9]) to prove (4.11).

Without loss of generality we assume that each U_p, $p \in \mathrm{zero}(V)$, is an open ball around p with radius $4a$.

Let $\gamma : \mathbf{R} \to [0,1]$ be a smooth function such that $\gamma(z) = 1$ if $|z| \leq a$, and that $\gamma(z) = 0$ if $|z| \geq 2a$.

For any $p \in \mathrm{zero}(V)$, $T > 0$, set

$$\alpha_{p,T} = \int_{U_p} \gamma(|\mathbf{y}|)^2 \exp\left(-T|\mathbf{y}A_p|^2\right) dv_{U_p}, \tag{4.34}$$

$$\rho_{p,T} = \frac{\gamma(|\mathbf{y}|)}{\sqrt{\alpha_{p,T}}} \exp\left(-\frac{T|\mathbf{y}A_p|^2}{2}\right) \cdot \rho_p. \tag{4.35}$$

Then $\rho_{p,T} \in \Omega^*(M)$ is a section of unit length with compact support contained in U_p.

Let E_T denote the direct sum of the vector spaces generated by $\rho_{p,T}$'s. Then E_T admits a $\mathbf{Z}_2$-graded decomposition

$$E_T = E_{T,\mathrm{even}} \oplus E_{T,\mathrm{odd}},$$

where $E_{T,\mathrm{even}}$ (resp. $E_{T,\mathrm{odd}}$) is the direct sum of the vector spaces generated by those $\rho_{p,T}$'s with $\det(A_p) > 0$ (resp. $\det(A_p) < 0$).

Let $E_T^{\perp}$ be the orthogonal complement to E_T in $\mathbf{H}^0(M)$. Then $\mathbf{H}^0(M)$ admits an orthogonal splitting

$$\mathbf{H}^0(M) = E_T \oplus E_T^{\perp}. \tag{4.36}$$

Let p_T, $p_T^{\perp}$ denote the orthogonal projections from $\mathbf{H}^0(M)$ to E_T, $E_T^{\perp}$ respectively.

Following Bismut and Lebeau [BL, Chap. 9], we decompose the Witten deformation operator D_T according to the splitting (4.36). That is, we define

$$D_{T,1} = p_T D_T p_T, \qquad D_{T,2} = p_T D_T p_T^{\perp},$$

$$D_{T,3} = p_T^{\perp} D_T p_T, \qquad D_{T,4} = p_T^{\perp} D_T p_T^{\perp}. \tag{4.37}$$

Let $\mathbf{H}^1(M)$ denote the first Sobolev space with respect to a (fixed) first Sobolev norm on $\Omega^*(M)$.

We now state a crucial result which will be proved in the next section.

Proposition 4.10 *There exists a constant $T_0 > 0$ such that*

(i) for any $T \geq T_0$ and $0 \leq u \leq 1$, the operator

$$D_T(u) = D_{T,1} + D_{T,4} + u\,(D_{T,2} + D_{T,3}) : \mathbf{H}^1(M) \longrightarrow \mathbf{H}^0(M) \tag{4.38}$$

is Fredholm;

(ii) the operator $D_{T,4} : E_T^{\perp} \cap \mathbf{H}^1(M) \to E_T^{\perp}$ is invertible.

Proof of the Poincaré-Hopf formula (4.11). By (4.18), Proposition 4.10 and the homotopy invariance of the index of Fredholm operators, one deduces that for $T \geq T_0$,

$$\chi(M) = \operatorname{ind}\left(D_T : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)\right)$$

$$= \operatorname{ind}\left(D_T(0) : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)\right)$$

$$= \operatorname{ind}\left(D_{T,1} : E_{T,\text{even}} \to E_{T,\text{odd}}\right)$$

$$= \sum_{p \in \text{zero}(V)} \operatorname{sgn}(\det(A_p)).$$

□

4.7 Some Estimates for $D_{T,i}$'s, $2 \leq i \leq 4$

In this section, we prove Proposition 4.10. The proof will be based on certain estimates for the decomposed operators $D_{T,i}$'s, $2 \leq i \leq 4$. These estimates are much simpler than the corresponding estimates in [BL, Chap. 9].

We first state the estimates for $D_{T,2}$ and $D_{T,3}$ as follows.

Proposition 4.11 *There exists constant $T_0 > 0$ such that for any $s \in E_T^{\perp} \cap \mathbf{H}^1(M)$, $s' \in E_T$ and $T \geq T_0$, one has*

$$\|D_{T,2}s\|_0 \leq \frac{\|s\|_0}{T},$$

$$\|D_{T,3}s'\|_0 \leq \frac{\|s'\|_0}{T}. \tag{4.39}$$

Proof. It is easy to see that $D_{T,3}$ is the formal adjoint of $D_{T,2}$. Thus one needs only to prove the first estimate in (4.39).

Since each $\rho_{p,T}$, $p \in \text{zero}(V)$, has support in U_p, by (4.35) and Proposition 4.9 one deduces that for any $s \in E_T^{\perp} \cap \mathbf{H}^1(M)$,

$$D_{T,2}s = \sum_{p \in \text{zero}(V)} \rho_{p,T} \int_{U_p} \langle \rho_{p,T}, D_T s \rangle \, dv_{U_p}$$

$$= \sum_{p \in \mathrm{zero}(V)} \rho_{p,T} \int_{U_p} \left\langle D_T \left(\frac{\gamma(|\mathbf{y}|)}{\sqrt{\alpha_{p,T}}} e^{-T|\mathbf{y} A_p|^2/2} \rho_p \right), s \right\rangle dv_{U_p}$$

$$= \sum_{p \in \mathrm{zero}(V)} \rho_{p,T} \int_{U_p} \left\langle \frac{c(d\gamma(|\mathbf{y}|))}{\sqrt{\alpha_{p,T}}} e^{-T|\mathbf{y} A_p|^2/2} \rho_p, s \right\rangle dv_{U_p}. \tag{4.40}$$

Now since γ equals to one in an open neighborhood around zero(V), $d\gamma$ vanishes on this open neighborhood. Thus by (4.40), one deduces easily that there exist constants $T_0 > 0$, $C_1 > 0$, $C_2 > 0$ such that when $T \geq T_0$, for any $s \in E_T^{\perp} \cap \mathbf{H}^1(M)$,

$$\|D_{T,2} s\|_0 \leq C_1 T^{n/2} \exp(-C_2 T) \|s\|_0, \tag{4.41}$$

from which the first inequality in (4.39) follows easily. □

By Proposition 4.11, one sees that both $D_{T,2}$ and $D_{T,3}$ are *compact* operators† mapping from $\mathbf{H}^1(M)$ to $\mathbf{H}^0(M)$. Thus one gets the first part of Proposition 4.10.

To get the second part of Proposition 4.10, one needs only to show that there exist constants $T_0 > 0$, $C_3 > 0$ such that for any $T \geq T_0$ and $s \in E_T^{\perp} \cap \mathbf{H}^1(M)$,

$$\|D_{T,4} s\|_0 \geq C_3 \|s\|_0.$$

Now since for $s \in E_T^{\perp} \cap \mathbf{H}^1(M)$ one has

$$D_T s = D_{T,2} s + D_{T,4} s,$$

by Proposition 4.11, one needs only to show that for some constant $C_4 > 0$,

$$\|D_T s\|_0 \geq C_4 \|s\|_0,$$

when $T > 0$ is large enough.

Proposition 4.12 *There exist constants $T_0 > 0$ and $C > 0$ such that for any $s \in E_T^{\perp} \cap \mathbf{H}^1(M)$ and $T \geq T_0$,*

$$\|D_T s\|_0 \geq C\sqrt{T} \|s\|_0. \tag{4.42}$$

†In fact, they are finite rank linear bounded operators.

Proof. For any $0 < b \le 4a$, we denote $U_p(b)$, $p \in \mathrm{zero}(V)$, the open ball around p with radius b.

Following Bismut and Lebeau [BL, Chap. 9], we will prove Proposition 4.12 in the following three steps:

(i) Step 1. We assume $\mathrm{Supp}(s) \subset \cup_{p\in \mathrm{zero}(V)} U_p(4a)$;

(ii) Step 2. We assume $\mathrm{Supp}(s) \subset M \setminus \cup_{p\in \mathrm{zero}(V)} U_p(2a)$;

(iii) Step 3. We prove the general case.

We now start to prove Proposition 4.12 step by step.

Step 1. We suppose $\mathrm{Supp}(s) \in \cup_{p\in \mathrm{zero}(V)} U_p(4a)$. Then we can assume as well that we are in a union of Euclidean spaces E_p's containing U_p's, $p \in \mathrm{zero}(V)$, and can thus apply the results in Section 4.5.

Thus, for any $T > 0$, $p \in \mathrm{zero}(V)$, set

$$\rho'_{p,T} = \left(\frac{T}{\pi}\right)^{n/4} \sqrt{|\det(A_p)|} \exp\left(\frac{-T|\mathbf{y}A_p|^2}{2}\right) \cdot \rho_p. \tag{4.43}$$

And for any section s verifying $\mathrm{Supp}(s) \in \cup_{p\in \mathrm{zero}(V)} U_p(4a)$, set

$$p'_T s = \sum_{p\in \mathrm{zero}(V)} \rho'_{p,T} \int_{\mathrm{E}_p} \langle \rho'_{p,T}, s\rangle \, dv_{\mathrm{E}_p}. \tag{4.44}$$

Then p'_T is the orthogonal projection from $\oplus_{p\in \mathrm{zero}(V)} \mathbf{H}^0(\mathrm{E}_p)$ to the finite dimensional vector space generated by $\rho'_{p,T}$, $p \in \mathrm{zero}(V)$.

Since $p_T s = 0$, we can rewrite $p'_T s$ as

$$p'_T s = \sum_{p\in \mathrm{zero}(V)} \rho'_{p,T}$$

$$\int_{\mathrm{E}_p} \left\langle (1-\gamma(|\mathbf{y}|)) \left(\frac{T}{\pi}\right)^{n/4} \sqrt{|\det(A_p)|} \exp\left(\frac{-T|\mathbf{y}A_p|^2}{2}\right) \cdot \rho_p, s \right\rangle dv_{\mathrm{E}_p}. \tag{4.45}$$

As γ equals to one near each p, by (4.45) there exists $C_5 > 0$ such that when $T \ge 1$,

$$\|p'_T s\|_0^2 \le \frac{C_5}{\sqrt{T}} \|s\|_0^2. \tag{4.46}$$

By Proposition 4.9 and (4.44), we know that

$$D_T p'_T s = 0.$$

Moreover, by (4.46) and Proposition 4.9, there exist constants $C_6 > 0$, $C_7 > 0$ such that

$$\|D_T s\|_0^2 = \|D_T (s - p'_T s)\|_0^2 \geq C_6 T \|s - p'_T s\|_0^2$$

$$\geq \frac{C_6 T}{2} \|s\|_0^2 - C_7 \sqrt{T} \|s\|_0^2,$$

from which one sees directly that there exists $T_1 > 0$ such that for any $T \geq T_1$,

$$\|D_T s\|_0 \geq \frac{\sqrt{C_6 T}}{2} \|s\|_0. \tag{4.47}$$

Step 2. Since now $\mathrm{Supp}(s) \subset M \setminus \cup_{p \in \mathrm{zero}(V)} U_p(2a)$, one can proceed as in the proof of Proposition 4.7 to find constants $T_2 > 0$ and $C_8 > 0$, such that for any $T \geq T_2$,

$$\|D_T s\|_0 \geq C_8 \sqrt{T} \|s\|_0. \tag{4.48}$$

Step 3. Let $\widetilde{\gamma} \in C^\infty(M)$ be defined such that on each U_p, $p \in \mathrm{zero}(V)$, $\widetilde{\gamma}(\mathbf{y}) = \gamma(|\mathbf{y}|/2)$, and that $\widetilde{\gamma}|_{M \setminus \cup_{p \in \mathrm{zero}(V)} U_p(4a)} = 0$.

Now for any $s \in E_T^\perp \cap \mathbf{H}^1(M)$, one verifies easily that each

$$\widetilde{\gamma} s \in E_T^\perp \cap \mathbf{H}^1(M).$$

Thus, by the results in Steps 1 and 2, one deduces that there exists $C_9 > 0$ such that for any $T \geq T_1 + T_2$,

$$\|D_T s\|_0 \geq \frac{1}{2} \left(\|(1 - \widetilde{\gamma}) D_T s\|_0 + \|\widetilde{\gamma} D_T s\|_0 \right)$$

$$= \frac{1}{2} \left(\|D_T((1 - \widetilde{\gamma}) s) + [D, \widetilde{\gamma}] s\|_0 + \|D_T(\widetilde{\gamma} s) + [\widetilde{\gamma}, D] s\|_0 \right)$$

$$\geq \frac{\sqrt{T}}{2} \left(C_8 \|(1 - \widetilde{\gamma}) s\|_0 + \sqrt{C_6} \|\widetilde{\gamma} s\|_0 \right) - C_9 \|s\|_0$$

$$\geq C_{10} \sqrt{T} \|s\|_0 - C_9 \|s\|_0,$$

where $C_{10} = \min\{\sqrt{C_6}/2, C_8/2\}$, which completes the proof of Proposition 4.12. □

The proof of Proposition 4.10 is thus also completed. □

4.8 An Alternate Analytic Proof

We observe that by Proposition 4.7, in the simplest case where V has no zeros on M, the Poincaré-Hopf formula is a direct consequence of Witten's deformation.

On the other hand, a paper of Atiyah [A] contains another analytic proof of this simple statement, which we recall as follows.

Let V be a nowhere zero vector field on M. Without loss of generality we assume that $|V| \equiv 1$ on M.

From (4.13) and (4.16), one verifies directly that

$$\widehat{c}(V)\,(d+d^*)\,\widehat{c}(V) = -\,(d+d^*) + \widehat{c}(V)\sum_{i=1}^{n} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} V\right). \tag{4.49}$$

From (4.9), (4.49) and the homotopy invariance of the index for elliptic differential operators, one deduces that

$$\chi(M) = \mathrm{ind}\left(d+d^* : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M)\right)$$

$$= \mathrm{ind}\left(\widehat{c}(V)\,(d+d^*)\,\widehat{c}(V) : \Omega^{\mathrm{odd}}(M) \to \Omega^{\mathrm{even}}(M)\right)$$

$$= \mathrm{ind}\left(-\,(d+d^*) + \widehat{c}(V)\sum_{i=1}^{n} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} V\right) : \Omega^{\mathrm{odd}}(M) \to \Omega^{\mathrm{even}}(M)\right)$$

$$= \mathrm{ind}\left(d+d^* : \Omega^{\mathrm{odd}}(M) \to \Omega^{\mathrm{even}}(M)\right)$$

$$= -\chi(M),$$

from which one gets the desired equality $\chi(M) = 0$.

In [Z], an alternate analytic proof of the Poincaré-Hopf index formula is given by extending the above idea of Atiyah to manifolds with boundary. The proof in [Z] works also for the case where the (isolated) zeros of the vector field may be *degenerate*, i.e., the vector field in question may not be a transversal section of TM. This is different from Witten's proof presented in this section.

4.9 References

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Chapter 5

Morse Inequalities: an Analytic Proof

In this chapter, we present Witten's analytic proof of Morse inequalities by refining some of the arguments in Chapter 4.

Witten's original paper [W1] has been very influential in various aspects in topology, geometry and mathematical physics. We will mention some of them in Section 5.7.

We recommend the book of Milnor [Mi] for a beautiful account of some of the classical aspects of Morse theory.

As in Chapter 4, we will work with real coefficients in this chapter.

5.1 Review of Morse Inequalities

Let M be an n-dimensional closed oriented manifold. Let $f \in C^\infty(M)$ be a smooth function on M. A point $x \in M$ is called a **critical point** of f if

$$df(x) = 0.$$

If $x \in M$ is a critical point of f, then we say x is **nondegenerate** if the Hessian of f at x is non-singular, i.e.,

$$\det(\mathrm{Hess}_f(x)) \neq 0.$$

It is easy to verify that every nondegenerate critical point $x \in M$ of f is isolated, that is, there is no other critical point of f in a sufficiently small open neighborhood of $x \in M$.

A smooth function on M is called a **Morse function** if all the critical points of this function are nondegenerate. It is well-known (cf. [Mi]) that

there always exists a Morse function on M. Clearly, a Morse function on a closed manifold has only a finite number of critical points.

From now on we assume f is a Morse function on M.

The following **Morse lemma** (cf. [Mi]) is important in many aspects of the theory of Morse functions.

Lemma 5.1 *For any critical point $x \in M$ of the Morse function f, there is an open neighborhood U_x of x and an oriented coordinate system $\mathbf{y} = (y^1, \ldots, y^n)$ such that on U_x, one has*

$$f(\mathbf{y}) = f(x) - \frac{1}{2}\left(y^1\right)^2 - \cdots - \frac{1}{2}\left(y^{n_f(x)}\right)^2 + \frac{1}{2}\left(y^{n_f(x)+1}\right)^2 + \cdots + \frac{1}{2}\left(y^n\right)^2. \tag{5.1}$$

We call the integer $n_f(x)$ the **Morse index** of f at x. Also, for later use, we assume that for any two different critical points x, $y \in M$ of f, $U_x \cap U_y = \emptyset$.

Now for any integer i such that $0 \leq i \leq n$, let β_i denote the i-th Betti number $\dim H^i_{\mathrm{dR}}(M; \mathbf{R})$. Let m_i denote the number of critical points $x \in M$ of f such that $n_f(x) = i$.

The **Morse inequalities**, for which an analytic proof will be given in this chapter, can be stated as follows.

Theorem 5.2 *(i)* **Weak Morse inequalities**: *For any integer i such that $0 \leq i \leq n$, one has*

$$\beta_i \leq m_i. \tag{5.2}$$

(ii) **Strong Morse inequalities**: *For any integer i such that $0 \leq i \leq n$, one has*

$$\beta_i - \beta_{i-1} + \cdots + (-1)^i\beta_0 \leq m_i - m_{i-1} + \cdots + (-1)^i m_0. \tag{5.3}$$

Moreover,

$$\beta_n - \beta_{n-1} + \cdots + (-1)^n\beta_0 = m_n - m_{n-1} + \cdots + (-1)^n m_0. \tag{5.4}$$

Clearly, (5.2) is a consequence of (5.3).*

*In fact, one can apply (5.3) twice to i and $i-1$ respectively, and then take sum to get (5.2).

We refer to the book [Mi] for a topological proof of this result. In the rest of this chapter, we will present an analytic proof of it by following an idea of Witten [W1].

5.2 Witten Deformation

Recall from Section 1.1 the definition of the de Rham complex

$$(\Omega^*(M), d) : 0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim M}(M) \longrightarrow 0.$$

Given the Morse function f, inspired by considerations in physics, Witten [W1] suggested to deform the exterior differential operator d as follows: for any $T \in \mathbf{R}$, set

$$d_{Tf} = e^{-Tf} d e^{Tf}. \tag{5.5}$$

Since $d^2 = 0$, from (5.5) one has

$$d_{Tf}^2 = 0. \tag{5.6}$$

Thus, one can deform the de Rham complex $(\Omega^*(M), d)$ to the complex $(\Omega^*(M), d_{Tf})$ defined by

$$(\Omega^*(M), d_{Tf}) : 0 \longrightarrow \Omega^0(M) \xrightarrow{d_{Tf}} \Omega^1(M) \xrightarrow{d_{Tf}} \cdots \xrightarrow{d_{Tf}} \Omega^{\dim M}(M) \longrightarrow 0.$$

Let

$$H^*_{Tf,\mathrm{dR}}(M; \mathbf{R}) = \frac{\ker d_{Tf}}{\operatorname{Im} d_{Tf}}$$

be the corresponding cohomology, with the $\mathbf{Z}$-grading given by

$$H^*_{Tf,\mathrm{dR}}(M; \mathbf{R}) = \bigoplus_{i=0}^{n} H^i_{Tf,\mathrm{dR}}(M; \mathbf{R}),$$

where for each integer i such that $0 \leq i \leq n$,

$$H^i_{Tf,\mathrm{dR}}(M; \mathbf{R}) = \frac{\ker d_{Tf}|_{\Omega^i(M)}}{\operatorname{Im} d_{Tf}|_{\Omega^{i-1}(M)}}.$$

The first important result for the Witten deformation is as follows.

Proposition 5.3 *For any integer i such that $0 \leq i \leq n$,*

$$\dim H^i_{Tf,\mathrm{dR}}(M;\mathbf{R}) = \dim H^i_{\mathrm{dR}}(M;\mathbf{R}).$$

Proof. For any $\alpha \in \Omega^i(M)$ such that $d\alpha = 0$, one verifies that

$$d_{Tf}(e^{-Tf}\alpha) = 0,$$

while for any $\beta \in \Omega^{i-1}(M)$, one has

$$e^{-Tf}d\beta = d_{Tf}(e^{-Tf}\beta).$$

Thus, the map

$$\alpha \in \Omega^i(M) \mapsto e^{-Tf}\alpha \in \Omega^i(M)$$

induces a well-defined homomorphism from $H^i_{\mathrm{dR}}(M;\mathbf{R})$ to $H^i_{Tf,\mathrm{dR}}(M;\mathbf{R})$.

Similarly, one sees easily that the map

$$\alpha \in \Omega^i(M) \mapsto e^{Tf}\alpha \in \Omega^i(M)$$

induces a well-defined homomorphism from $H^i_{\mathrm{dR}}(M;\mathbf{R})$ to $H^i_{Tf,\mathrm{dR}}(M;\mathbf{R})$.

It is now easy to verify that these two induced homomorphisms on cohomologies are in fact isomorphisms each of which is the inverse of the other one. □

5.3 Hodge Theorem for $(\Omega^*(M), d_{Tf})$

Let g^{TM} be a metric on TM. Recall that the Hodge theorem for the de Rham complex $(\Omega^*(M), d)$ has been reviewed in Section 4.1.

Since $T \in \mathbf{R}$, by (4.4) one deduces that for any $\alpha,\ \beta \in \Omega^*(M)$,

$$\langle d_{Tf}\alpha, \beta\rangle = \left\langle e^{-Tf}de^{Tf}\alpha, \beta\right\rangle = \left\langle \alpha, e^{Tf}d^*e^{-Tf}\beta\right\rangle.$$

Thus,

$$d^*_{Tf} := e^{Tf}d^*e^{-Tf} \tag{5.7}$$

is the formal adjoint of d_{Tf}.

Recall that $D = d + d^*$. For any $T \geq 0$, set

$$D_{Tf} = d_{Tf} + d^*_{Tf}, \tag{5.8}$$

$$\square_{Tf} = D^2_{Tf} = d_{Tf}d^*_{Tf} + d^*_{Tf}d_{Tf}. \tag{5.9}$$

By (5.5) and (5.7), one sees that $\Box_{Tf}$ preserves each $\Omega^i(M)$, $0 \le i \le n$. Moreover, one can well establish the Hodge theorem for the complex $(\Omega^*(M), d_{Tf})$, a consequence of which implies that for any integer i such that $0 \le i \le n$,

$$\dim\left(\ker \Box_{Tf}|_{\Omega^i(M)}\right) = \dim H^i_{Tf,\mathrm{dR}}(M;\mathbf{R}) = \dim H^i_{\mathrm{dR}}(M;\mathbf{R}), \qquad (5.10)$$

where the last equality follows from Proposition 5.3.

From (5.10), one sees that to obtain the information about the β_i's, one may take $T \to +\infty$ and study the behaviour of $\Box_{Tf}$ under the limit.

5.4 Behaviour of $\Box_{Tf}$ Near the Critical Points of f

Without loss of generality, we assume that on the open neighborhood U_x of a critical point $x \in M$ of f, with the coordinate system $\mathbf{y} = (y^1, \ldots, y^n)$ which are defined in Section 5.1, one has

$$g^{TM} = \left(dy^1\right)^2 + \cdots + \left(dy^n\right)^2. \qquad (5.11)$$

From (4.14)-(4.16), (5.5), (5.7) and (5.8), one verifies directly that

$$d_{Tf} = d + Tdf\wedge, \qquad d^*_{Tf} = d^* + Ti_{df}$$

and

$$D_{Tf} = D + T\widehat{c}(df), \qquad (5.12)$$

where we identify df with its corresponding element in $\Gamma(TM)$ determined by g^{TM}.

Clearly, (5.12) is a special case of the deformation (4.17) in Section 4.3. However, the deformation operator in (5.12) has the advantage that the square of it preserves the $\mathbf{Z}$-grading of $\Omega^*(M)$, while the square of the deformation operator in (4.17) only preserves the $\mathbf{Z}_2$-grading of $\Omega^*(M)$, in general.

Now by the Morse lemma 5.1, one verifies that on each U_x, one has

$$df(x) = -y^1dy^1 - \cdots - y^{n_f(x)}dy^{n_f(x)} + y^{n_f(x)+1}dy^{n_f(x)+1} + \cdots + y^ndy^n. \qquad (5.13)$$

Let $e_i = \frac{\partial}{\partial y^i}$, $1 \le i \le n$, be the oriented orthonormal basis of TU_x.

By (5.11)-(5.13) and the Bochner type formula (4.19), one deduces that on each U_x,

$$\square_{Tf} = -\sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)^2 - nT + T^2|\mathbf{y}|^2$$

$$+T\sum_{i=1}^{n_f(x)}(1-c(e_i)\widehat{c}(e_i)) + T\sum_{i=n_f(x)+1}^{n}(1+c(e_i)\widehat{c}(e_i))$$

$$= -\sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)^2 - nT + T^2|\mathbf{y}|^2$$

$$+2T\left(\sum_{i=1}^{n_f(x)} i_{e_i}e_i^*\wedge + \sum_{i=n_f(x)+1}^{n} e_i^*\wedge i_{e_i}\right). \tag{5.14}$$

It is easy to verify that the linear operator

$$\sum_{i=1}^{n_f(x)} i_{e_i}e_i^*\wedge + \sum_{i=n_f(x)+1}^{n} e_i^*\wedge i_{e_i}$$

is nonnegative, with the kernel being one dimensional and generated by

$$dy^1\wedge\cdots\wedge dy^{n_f(x)}.$$

One then gets the following $\mathbf{Z}$-graded refinement of Proposition 4.9 in the current situation.

Proposition 5.4 *For any $T>0$, the operator*

$$-\sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)^2 - nT + T^2|\mathbf{y}|^2 + 2T\left(\sum_{i=1}^{n_f(x)} i_{e_i}e_i^*\wedge + \sum_{i=n_f(x)+1}^{n} e_i^*\wedge i_{e_i}\right)$$

acting on $\Gamma(\Lambda^*(\mathrm{E}_n^*))$ is nonnegative. Its kernel is of dimension one and is generated by

$$\exp\left(\frac{-T|\mathbf{y}|^2}{2}\right)\cdot dy^1\wedge\cdots\wedge dy^{n_f(x)}.$$

Moreover, all the nonzero eigenvalues of this operator are greater than CT for some fixed constant $C > 0$.

5.5 Proof of Morse Inequalities

Recall that in the proof of the Poincaré-Hopf index formula in Section 4.6, we have used the deformation (4.38) to reduce the proof to a finite dimensional situation. However, if we would apply this deformation to the operator D_{Tf} now, we would see that the Laplacians of the deformed operators only preserve the $\mathbf{Z}_2$-grading of $\Omega^*(M)$, not the required $\mathbf{Z}$-grading nature. Thus, one should deal with more refined arguments.

Following Witten [W1], we will instead prove the following result, from which the Morse inequalities will follow.

Proposition 5.5 *For any $c > 0$, there exists $T_0 > 0$ such that when $T \geq T_0$, the number of eigenvalues in $[0, c]$ of $\square_{Tf}|_{\Omega^i(M)}$, $0 \leq i \leq n$, equals to m_i.*

Proposition 5.5 will be proved in the next section.

We now prove the Morse inequalities by using Proposition 5.5.

For any integer i such that $0 \leq i \leq n$, let

$$\mathrm{F}_{Tf,i}^{[0,c]} \subset \Omega^*(M)$$

denote the m_i dimensional vector space generated by the eigenspaces of $\square_{Tf}|_{\Omega^i(M)}$ associated with eigenvalues in $[0, c]$.

Since

$$d_{Tf}\square_{Tf} = \square_{Tf}d_{Tf} = d_{Tf}d_{Tf}^*d_{Tf}$$

and

$$d_{Tf}^*\square_{Tf} = \square_{Tf}d_{Tf}^* = d_{Tf}^*d_{Tf}d_{Tf}^*,$$

one sees that d_{Tf} (resp. d_{Tf}^*) maps each $\mathrm{F}_{Tf,i}^{[0,c]}$ to $\mathrm{F}_{Tf,i+1}^{[0,c]}$ (resp. $\mathrm{F}_{Tf,i-1}^{[0,c]}$). Thus, one has the following finite dimensional subcomplex of $(\Omega^*(M), d_{Tf})$:

$$\left(\mathrm{F}_{Tf}^{[0,c]}, d_{Tf}\right) : 0 \longrightarrow \mathrm{F}_{Tf,0}^{[0,c]} \xrightarrow{d_{Tf}} \mathrm{F}_{Tf,1}^{[0,c]} \xrightarrow{d_{Tf}} \cdots \xrightarrow{d_{Tf}} \mathrm{F}_{Tf,n}^{[0,c]} \longrightarrow 0. \tag{5.15}$$

Moreover, one can prove a Hodge decomposition theorem for this finite dimensional complex (or one can just apply the restriction of the Hodge

decomposition theorem for $(\Omega^*(M), d_{Tf})$ to this finite dimensional complex). In particular, for any integer i such that $0 \leq i \leq n$,

$$\beta_{Tf,i}^{[0,c]} := \dim \left(\frac{\ker d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}}}{\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i-1}^{[0,c]}}} \right)$$

equals to $\dim(\ker \square_{Tf}|_{\Omega^i(M)})$, which in turn equals to β_i by (5.10). By Proposition 5.5, this implies the *weak Morse inequalities.*

To prove the strong Morse inequalities, we examine the following decompositions obtained from the complex (5.15): for any integer i such that $0 \leq i \leq n$,

$$\dim \mathrm{F}_{Tf,i}^{[0,c]} = \dim \left(\ker d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}} \right) + \dim \left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}} \right)$$

$$= \dim \left(\frac{\ker d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}}}{\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i-1}^{[0,c]}}} \right) + \dim \left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i-1}^{[0,c]}} \right) + \left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}} \right). \tag{5.16}$$

From Proposition 5.5 and (5.16), one deduces easily that for any integer i such that $0 \leq i \leq n$,

$$\sum_{j=0}^{i} (-1)^j m_{i-j} = \sum_{j=0}^{i} (-1)^j \left(\beta_{i-j} + \dim \left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i-j-1}^{[0,c]}} \right) \right.$$

$$\left. + \dim \left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i-j}^{[0,c]}} \right) \right)$$

$$= \sum_{j=0}^{i} (-1)^j \beta_{i-j} + \dim \left(\operatorname{Im} d_{Tf}|_{\mathrm{F}_{Tf,i}^{[0,c]}} \right),$$

from which the *strong Morse inequalities* follows. In particular, we see that when $i = n$, the equality (5.4) holds. □

Clearly, equality (5.4) is a special case of the Poincaré-Hopf index formula proved in Chapter 4.

5.6 Proof of Proposition 5.5

We will proceed as in Sections 4.6 and 4.7, which in turn rely on techniques developed in [BL, Chap. 9], to prove Proposition 5.5.

As in (4.34) and (4.35), in view of Proposition 5.4, for any $T > 0$ and critical point $x \in M$ of f, set

$$\alpha_{x,T} = \int_{U_x} \gamma(|\mathbf{y}|)^2 \exp\left(-T|\mathbf{y}|^2\right) dy^1 \wedge \cdots \wedge dy^n,$$

$$\rho_{x,T} = \frac{\gamma(|\mathbf{y}|)}{\sqrt{\alpha_{x,T}}} \exp\left(-\frac{T|\mathbf{y}|^2}{2}\right) dy^1 \wedge \cdots \wedge dy^{n_f(x)}. \tag{5.17}$$

Then $\rho_{x,T} \in \Omega^{n_f(x)}(M)$ is of unit length with compact support contained in U_x.

Let E_T denote the direct sum of the vector spaces generated by $\rho_{x,T}$'s, where x runs through the set of critical points of f. Let $E_T^{\perp}$ be the orthogonal complement to E_T in $\mathbf{H}^0(M)$. Then $\mathbf{H}^0(M)$ admits an orthogonal splitting

$$\mathbf{H}^0(M) = E_T \oplus E_T^{\perp}. \tag{5.18}$$

Let p_T, $p_T^{\perp}$ denote the orthogonal projection operators from $\mathbf{H}^0(M)$ to E_T, $E_T^{\perp}$ respectively.

As in (4.37), we decompose the Witten deformed operator D_{Tf} by setting

$$D_{T,1} = p_T D_{Tf} p_T, \qquad D_{T,2} = p_T D_{Tf} p_T^{\perp},$$

$$D_{T,3} = p_T^{\perp} D_{Tf} p_T, \qquad D_{T,4} = p_T^{\perp} D_{Tf} p_T^{\perp}. \tag{5.19}$$

As in Section 4.7, the estimates summarized in the following proposition are crucial.

Proposition 5.6 *(i) For any $T > 0$,*

$$D_{T,1} = 0; \tag{5.20}$$

(ii) There exists constant $T_1 > 0$, such that for any $s \in E_T^{\perp} \cap \mathbf{H}^1(M)$, $s' \in E_T$ and $T \geq T_1$, one has

$$\|D_{T,2}s\|_0 \leq \frac{\|s\|_0}{T},$$

$$\|D_{T,3}s'\|_0 \leq \frac{\|s'\|_0}{T}; \tag{5.21}$$

(iii) There exist $T_2 > 0$ and $C > 0$ such that for any $s \in E_T^{\perp} \cap \mathbf{H}^1(M)$ and $T \geq T_2$,

$$\|D_{Tf}s\|_0 \geq C\sqrt{T}\|s\|_0. \tag{5.22}$$

Proof. (i) Let zero(df) denote the set of critical points of f. Then for any $s \in \mathbf{H}^0(M)$, one verifies directly that

$$p_T s = \sum_{x \in \mathrm{zero}(df)} \langle \rho_{x,T}, s\rangle_{\mathbf{H}^0(M)}\, \rho_{x,T}. \tag{5.23}$$

By (5.17) it is clear that for any $x \in \mathrm{zero}(df)$,

$$D_{Tf}\left(\langle \rho_{x,T}, s\rangle_{\mathbf{H}^0(M)}\rho_{x,T}\right) \in \Omega^{n_f(x)-1}(M) \oplus \Omega^{n_f(x)+1}(M) \tag{5.24}$$

has compact support in U_x. Thus, (5.20) follows.

(ii) This is a special case of Proposition 4.11.

(iii) This is a special case of Proposition 4.12.

The proof of Proposition 5.6 is completed. □

Remark 5.7 Similarly, one can show that the operator $D_{T,1}$ in Section 4.6 is also a zero operator. We did not make this explicit since this fact was not used there.

Now for any positive constant $c > 0$, let $E_T(c)$ denote the direct sum of eigenspaces of D_{Tf} associated with the eigenvalues lying in $[-c, c]$. Clearly, $E_T(c)$ is a finite dimensional subspace of $\mathbf{H}^0(M)$.

Let $P_T(c)$ denote the orthogonal projection operator from $\mathbf{H}^0(M)$ to $E_T(c)$.

Lemma 5.8 *There exist $C_1 > 0$, $T_3 > 0$ such that for any $T \geq T_3$ and any $\sigma \in E_T$,*

$$\|P_T(c)\sigma - \sigma\|_0 \leq \frac{C_1}{T}\|\sigma\|_0. \tag{5.25}$$

Proof. Let $\delta = \{\lambda \in \mathbf{C} : |\lambda| = c\}$ be the counter-clockwise oriented circle. By Proposition 5.6, one deduces that for any $\lambda \in \delta$, $T \geq T_1 + T_2$ and $s \in \mathbf{H}^1(M)$,

$$\|(\lambda - D_{Tf})\, s\|_0 \geq \frac{1}{2}\left\|\lambda p_T s - D_{T,2}p_T^{\perp} s\right\|_0$$

$$+\frac{1}{2}\left\|\lambda p_T^{\perp} s - D_{T,3} p_T s - D_{T,4} p_T^{\perp} s\right\|_0$$

$$\geq \frac{1}{2}\left(\left(c - \frac{1}{T}\right)\|p_T s\|_0 + \left(C\sqrt{T} - c - \frac{1}{T}\right)\left\|p_T^{\perp} s\right\|_0\right). \tag{5.26}$$

By (5.26), one sees that there exist $T_4 > T_1 + T_2$ and $C_2 > 0$ such that for any $T \geq T_4$ and $s \in \mathbf{H}^1(M)$,

$$\|(\lambda - D_{Tf})\, s\|_0 \geq C_2\|s\|_0. \tag{5.27}$$

Thus, for any $T \geq T_4$ and $\lambda \in \delta$,

$$\lambda - D_{Tf} : \mathbf{H}^1(M) \to \mathbf{H}^0(M)$$

is invertible.

Thus, the resolvent $(\lambda - D_{Tf})^{-1}$ is well-defined.

By the basic spectral theorem in operator theory (cf. [D]), one has

$$P_T(c)\sigma - \sigma = \frac{1}{2\pi\sqrt{-1}}\int_\delta \left((\lambda - D_{Tf})^{-1} - \lambda^{-1}\right)\sigma d\lambda. \tag{5.28}$$

Now one verifies directly by Proposition 5.6(i) that

$$\left((\lambda - D_{Tf})^{-1} - \lambda^{-1}\right)\sigma = \lambda^{-1}(\lambda - D_{Tf})^{-1} D_{T,3}\sigma. \tag{5.29}$$

From Proposition 5.6(ii) and (5.27), one then deduces that for any $T \geq T_4$ and $\sigma \in E_T$,

$$\left\|(\lambda - D_{Tf})^{-1} D_{T,3}\sigma\right\|_0 \leq C_2^{-1}\|D_{T,3}\sigma\|_0 \leq \frac{1}{C_2 T}\|\sigma\|_0. \tag{5.30}$$

From (5.28)-(5.30), one gets (5.25). □

Remark 5.9 Though there have been used complex numbers in the above proof (by which one needs to complexify the spaces and extend the operators accordingly, though this was not stated explicitly in the proof), one can well stay in the real coefficient category by working with the real part of the right hand side of (5.28). We leave these to the interested reader.

Proof of Proposition 5.5. By applying Lemma 5.8 to the $\rho_{x,T}$'s for $x \in \mathrm{zero}(df)$, one sees easily that when T is large enough, the $P_T(c)\rho_{x,T}$'s for $x \in \mathrm{zero}(df)$ are linearly independent. Thus, there exists $T_5 > 0$ such that when $T \geq T_5$,

$$\dim E_T(c) \geq \dim E_T. \tag{5.31}$$

Now if $\dim E_T(c) > \dim E_T$, then there should exist a nonzero element $s \in E_T(c)$ such that s is perpendicular to $P_T(c)E_T$. That is,

$$\langle s, P_T(c)\rho_{x,T}\rangle_{\mathbf{H}^0(M)} = 0 \tag{5.32}$$

for any $x \in \mathrm{zero}(df)$.

From (5.23) and (5.32), one deduces that

$$\begin{aligned} p_T s = & \sum_{x\in \mathrm{zero}(df)} \langle s, \rho_{x,T}\rangle_{\mathbf{H}^0(M)}\, \rho_{x,T} \\ & - \sum_{x\in \mathrm{zero}(df)} \langle s, P_T(c)\rho_{x,T}\rangle_{\mathbf{H}^0(M)}\, P_T(c)\rho_{x,T} \\ = & \sum_{x\in \mathrm{zero}(df)} \langle s, \rho_{x,T}\rangle_{\mathbf{H}^0(M)}\, (\rho_{x,T} - P_T(c)\rho_{x,T}) \\ & + \sum_{x\in \mathrm{zero}(df)} \langle s, \rho_{x,T} - P_T(c)\rho_{x,T}\rangle_{\mathbf{H}^0(M)}\, P_T(c)\rho_{x,T}. \end{aligned} \tag{5.33}$$

By (5.33) and Lemma 5.8, there exists $C_3 > 0$ such that when $T \geq T_5$,

$$\|p_T s\|_0 \leq \frac{C_3}{T}\|s\|_0. \tag{5.34}$$

Thus, there exists a constant $C_4 > 0$ such that when $T > 0$ is large enough,

$$\left\|p_T^{\perp} s\right\|_0 \geq \|s\|_0 - \|p_T s\|_0 \geq C_4\|s\|_0. \tag{5.35}$$

From (5.35) and Proposition 5.6, one sees that when $T > 0$ is large enough,

$$CC_4\sqrt{T}\|s\|_0 \leq \left\|D_{Tf}p_T^{\perp}s\right\|_0 = \|D_{Tf}s - D_{Tf}p_Ts\|_0$$

$$= \|D_{Tf}s - D_{T,3}s\|_0 \leq \|D_{Tf}s\|_0 + \|D_{T,3}s\|_0$$

$$\leq \|D_{Tf}s\|_0 + \frac{1}{T}\|s\|_0,$$

from which one gets

$$\|D_{Tf}s\|_0 \geq CC_4\sqrt{T}\|s\|_0 - \frac{1}{T}\|s\|_0.$$

Clearly, when $T > 0$ is large enough, this contradicts with the assumption that s is a nonzero element in $E_T(c)$.

Thus, one has

$$\dim E_T(c) = \dim E_T = \sum_{i=0}^{n} m_i. \tag{5.36}$$

Moreover, $E_T(c)$ is generated by $P_T(c)\rho_{x,T}$'s for all $x \in \mathrm{zero}(df)$.

Now in order to prove Proposition 5.5, for any integer i such that $0 \leq i \leq n$, denote by Q_i the orthogonal projection operator from $\mathbf{H}^0(M)$ onto the L^2-completion space of $\Omega^i(M)$. Since $\square_{Tf}$ preserves the $\mathbf{Z}$-grading of $\Omega^*(M)$, one sees that for any eigenvector s of D_{Tf} associated with an eigenvalue $\mu \in [-c, c]$,

$$\square_{Tf}Q_is = Q_i\square_{Tf}s = \mu^2 Q_is.$$

That is, $Q_is \in \Omega^i(M)$ is an eigenvector of $\square_{Tf}$ associated with eigenvalue μ^2.

Thus, in order to prove Proposition 5.5, one needs only to show that when $T > 0$ is large enough,

$$\dim Q_iE_T(c) = m_i. \tag{5.37}$$

To prove (5.37), one uses Lemma 5.8 to see that for any $x \in \mathrm{zero}(df)$,

$$\left\|Q_{n_f(x)}P_T(c)\rho_{x,T} - \rho_{x,T}\right\|_0 \leq \frac{C_1}{T}. \tag{5.38}$$

From (5.38), one sees that when $T > 0$ is large enough, theforms $Q_{n_f(x)}P_T(c)\rho_{x,T}$, $x \in \mathrm{zero}(df)$, are linearly independent. Thus, for each integer i between 0 and n,

$$\dim Q_i E_T(c) \geq m_i. \tag{5.39}$$

On the other hand, by (5.36) one has

$$\sum_{i=0}^{n} \dim Q_i E_T(c) \leq \dim E_T(c) = \sum_{i=0}^{n} m_i. \tag{5.40}$$

From (5.39) and (5.40), one gets (5.37).

The proof of Proposition 5.5 is completed. □

Remark 5.10 Since the constant $c > 0$ in Proposition 5.5 can be chosen arbitrarily small, one sees that when $T \to +\infty$, the eigenvalues in $[0, c]$ of $\square_{Tf}$ converge to zero.

5.7 Some Remarks and Comments

1). Witten's original paper [W1] was very influential in 1980's. Many rigorous accounts of the analytic proof of the Morse inequalities appeared right after the appearance of [W1]. Here we only mention the paper by Helffer-Sjöstrand [HS] which was based on semi-classical analysis and the paper by Bismut [B] where a proof by heat equation methods was developed. The later also contains an analytic treatment of Bott-Morse inequalities which hold when the critical points are only nondegenerate in the sense of Bott [Bo1].

2). Witten further suggested in [W1] that under some generic conditions, from the complex $(\mathrm{F}_{Tf}^{[0,c]}, d_{Tf})$ defined in (5.15) one can even recover the Thom-Smale complex (cf. [L]) associated to the Morse function f. Witten's this idea, which was proved rigorously in [HS] (Compare also with [BZ2, Sect. 6]), has a tremendous influence on the subsequent developments. For example, it is one of the sources for Floer's conception [F] of Floer homology (cf. [Bo2] for a nice informal account on these). In another direction, Bismut and Zhang [BZ1] used these ideas to give a heat kernel proof, as well as an extension to the case of general flat vector bundles, of the theorems of Cheeger [C] and Müller ([Mü1], [Mü2]) on relations between

the Ray-Singer analytic torsion [RS] and the Reidemeister torsion. Most recently, a far reaching generalization of the main results in [BZ1] and [BZ2] to the case of fibrations has been obtained by Bismut and Goette, see [BG1] and [BG2] for more details.

3). In a subsequent paper [W2], Witten also proposed certain holomorphic Morse inequalities for circle actions on Kähler manifolds. These holomorphic Morse inequalities were first proved rigorously by Mathai and Wu [MW] by a heat equation method for the case where the fixed point set of the circle action consists of isolated points. The paper [WZ] contains a proof by using the analytic arguments similar to what in this chapter. It also covers the case where the fixed point set of the circle action may be non-isolated.

4). The analytic localization methods described in Chapters 4 and 5, with necessary technical refinements if needed, are very useful for a wide range of problems in index theory (cf. [BL]). We hope to have shown that the basic ideas involved are in fact very simple.

5.8 References

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Chapter 6

Thom-Smale and Witten Complexes

In the previous chapter, we presented an analytic proof of Morse inequalities by using the Witten deformation of the de Rham complex. We also pointed out in Section 5.7 that in his seminal paper [Wi], Witten further suggested that the Thom-Smale complex associated to generic Morse functions can also be recovered from his deformation, and that Witten's this suggestion was first realized rigorously by Helffer and Sjöstrand [HS] by using semi-classical approximation methods. In this chapter we will examine this point of view of Witten by adapting a simpler treatment appearing first in the paper of Bismut and Zhang [BZ2].

Since this chapter is closely related to the previous one, we will make the same assumptions and use the same notation as in the last chapter.

6.1 The Thom-Smale Complex

Let $f \in C^\infty(M)$ be a Morse function on an n-dimensional closed oriented manifold M.

Let g^{TM} be a metric on TM, and let

$$\nabla f = (df)^* \in \Gamma(TM)$$

be the corresponding gradient vector field of f. Then the following differential equation defines a group of diffeomorphisms $(\psi_t)_{t\in\mathbf{R}}$ of M:

$$\frac{dy}{dt} = -\nabla f(y). \tag{6.1}$$

If $x \in \mathrm{zero}(\nabla f)$, set

$$W^u(x) = \left\{ y \in M : \lim_{t \to -\infty} \psi_t(y) = x \right\},$$

$$W^s(x) = \left\{ y \in M : \lim_{t \to +\infty} \psi_t(y) = x \right\}. \tag{6.2}$$

The cells $W^u(x)$ and $W^s(x)$ will be called the unstable and stable cells at x respectively.

We assume that the vector field ∇f verifies the **Smale transversality conditions** [S]. Namely, we suppose that for any $x,\ y \in \mathrm{zero}(\nabla f)$ with $x \neq y$, $W^u(x)$ and $W^s(y)$ intersect transversally. In particular, if $n_f(y) = n_f(x) - 1$, then $W^u(x) \cap W^s(y)$ consists of a finite set $\Gamma(x,y)$ of integral curves γ of the vector field $-\nabla f$, with $\gamma_{-\infty} = x$ and $\gamma_{+\infty} = y$, along which $W^u(x)$ and $W^s(y)$ intersect transversally.

By [S, Theorem A], given a Morse function f, there always exists a metric g^{TM} on TM such that ∇f verifies the transversality conditions.

We fix an orientation on each $W^u(x)$, $x \in \mathrm{zero}(\nabla f)$.

Let $x,\ y \in \mathrm{zero}(\nabla f)$ with $n_f(y) = n_f(x) - 1$.

Take $\gamma \in \Gamma(x,y)$. Then the tangent space $T_y W^u(y)$ is orthogonal to the tangent space $T_y W^s(y)$ and is oriented.

For any $t \in (-\infty, +\infty)$, the orthogonal space $T^{\perp}_{\gamma_t} W^s(y)$ to $T_{\gamma_t} W^s(y)$ in $T_{\gamma_t} M$ carries a natural orientation, which is induced from the orientation on $T_y W^u(y)$.

On the other hand, also for $t \in (-\infty, +\infty)$, the orthogonal space $T'_{\gamma_t} W^u(x)$ to $-\nabla f(\gamma_t)$ in $T_{\gamma_t} W^u(x)$ can be oriented in such a way that s is an oriented basis of $T'_{\gamma_t} W^u(x)$ if $(-\nabla f(\gamma_t), s)$ is an oriented basis of $T_{\gamma_t} W^u(x)$.

Since $W^u(x)$ and $W^s(y)$ are transversal along γ, for any $t \in (-\infty, +\infty)$, $T^{\perp}_{\gamma_t} W^s(y)$ and $T'_{\gamma_t} W^u(x)$ can be identified, and thus one can compare the induced orientations on them.

Set

$$n_\gamma(x,y) = 1 \quad \text{if the orientations are the same,}$$

$$= -1 \quad \text{if the orientations differ.} \tag{6.3}$$

If $x \in \mathrm{zero}(\nabla f)$, let $[W^u(x)]$ be the real line generated by $W^u(x)$. Set

$$C_*(W^u) = \bigoplus_{x \in \mathrm{zero}(\nabla f)} [W^u(x)],$$

$$C_i(W^u) = \bigoplus_{\substack{x \in \mathrm{zero}(\nabla f) \\ n_f(x)=i}} [W^u(x)]. \tag{6.4}$$

If $x \in \mathrm{zero}(\nabla f)$, set

$$\partial W^u(x) = \sum_{\substack{y \in \mathrm{zero}(\nabla f) \\ n_f(y)=n_f(x)-1}} \sum_{\gamma \in \Gamma(x,y)} n_\gamma(x,y) W^y(y). \tag{6.5}$$

Then ∂ maps $C_i(W^u)$ to $C_{i-1}(W^u)$.

The following basic result is due to Thom [T] and Smale [S].

Theorem 6.1 *$(C_*(W^u), \partial)$ is a chain complex. Moreover, we have a canonical identification between its $\mathbf{Z}$-graded homology group $H_*(C_*(W^u), \partial)$ and the $\mathbf{Z}$-graded singular homology group $H_*(M)$.*

If $x \in \mathrm{zero}(\nabla f)$, let $[W^u(x)]^*$ be the line dual to $[W^u(x)]$. Let $(C^*(W^u), \widetilde{\partial})$ be the complex which is dual to $(C_*(W^u), \partial)$. For any integer i such that $0 \le i \le n$, we have the identity

$$C^i(W^u) = \bigoplus_{\substack{x \in \mathrm{zero}(\nabla f) \\ n_f(x)=i}} [W^u(x)]^*. \tag{6.6}$$

Then by Theorem 6.1, one has the identification of the $\mathbf{Z}$-graded cohomology spaces

$$H^*\left(C^*(W^u), \widetilde{\partial}\right) \simeq H^*_{\mathrm{Sing}}(M). \tag{6.7}$$

6.2 The de Rham Map for Thom-Smale Complexes

We now assume that for any $x \in \mathrm{zero}(\nabla f)$, there exists a sufficiently small open neighborhood U_x of x and a coordinate system $\mathbf{y} = (y^1, ..., y^n)$ on U_x such that on U_x,

$$f(\mathbf{y}) = f(x) - \frac{1}{2}\left(y^1\right)^2 - \cdots - \frac{1}{2}\left(y^{n_f(x)}\right)^2 + \frac{1}{2}\left(y^{n_f(x)+1}\right)^2 + \cdots + \frac{1}{2}\left(y^n\right)^2,$$

$$g^{TM} = \left(dy^1\right)^2 + \cdots + \left(dy^n\right)^2 . \tag{6.8}$$

Certainly we can assume that for any $x,\ y \in \mathrm{zero}(\nabla f)$ with $x \neq y$, $U_x \cap U_y = \emptyset$.

We still assume that ∇f verifies the Smale transversality conditions.

By the Morse lemma 5.1 and by [S], given a Morse function f , there always exists a metric g^{TM} on TM verifying the above conditions.

We now state a result of Laudenbach [L, Prop. 2] which improves an old result of Rosenberg [R].

Proposition 6.2 *(i) If $x \in \mathrm{zero}(\nabla f)$, then the closure $\overline{W}^u(x)$ is an $n_f(x)$-dimensional submanifold of M with conical singularities;*

(ii) $\overline{W}^u(x) \setminus W^u(x)$ is stratified by unstable manifolds of critical points of index strictly less than $n_f(x)$.

We refer to the original paper [L] for the proof of Proposition 6.2.

By Part (i) of Proposition 6.2, one sees that one can integrate smooth forms over $\overline{W}^u(x)$'s, $x \in \mathrm{zero}(\nabla f)$.

If $x \in \mathrm{zero}(\nabla f)$, then the line $[W^u(x)]$ has a canonical non-zero section $W^u(x)$. Let $W^u(x)^* \in [W^u(x)]^*$ be dual to $W^u(x)$ so that

$$(W^u(x), W^u(x)^*) = 1.$$

If $\alpha \in \Omega^*(M)$, then the integral

$$W^u(x)^* \int_{\overline{W}^u(x)} \alpha$$

lies in $[W^u(x)]^*$. Clearly, if $\alpha \in \Omega^i(M)$, then $\int_{\overline{W}^u(x)} \alpha$ is non-zero only if $n_f(x) = i$.

Definition 6.3 Let P_∞ be the map

$$\alpha \in \Omega^*(M) \longrightarrow \sum_{x \in \mathrm{zero}(\nabla f)} [W^u(x)]^* \int_{\overline{W}^u(x)} \alpha \in C^*(W^u). \tag{6.9}$$

Theorem 6.4 (Laudenbach, cf. [BZ1, Theorem 2.9]) *The map P_∞ is a $\mathbf{Z}$-graded quasi-isomorphism between the de Rham complex $(\Omega^*(M), d)$ and*

the dual Thom-Smale complex $(C^*(W^u), \widetilde{\partial})$, which provides the canonical identification of the cohomology groups of both complexes.

The following particular formula, which shows that P_∞ is actually a chain homomorphism, follows in fact easily from (6.5), Proposition 6.2 and the Stokes formula,*

$$P_\infty d = \widetilde{\partial} P_\infty. \tag{6.10}$$

What Witten [Wi] suggested is that Theorem 6.4 can be recovered from the deformations (5.5) and (5.7) by letting $T \to +\infty$. This was first realized by Helffer-Sjöstrand [HS] by using semi-classical approximation methods. In the next sections we will present a treatment which is adapted from [BZ2, Section 6].

6.3 Witten's Instanton Complex and the Map e_T

Let $T_0 > 0$ be such that Proposition 5.5 holds for $c = 1$ and any $T \geq T_0$. From now on we always assume that $T \geq T_0$.

Recall from Section 5.5 that for any integer i such that $0 \leq i \leq n$,

$$\mathrm{F}^{[0,1]}_{Tf,i} \subset \Omega^i(M)$$

is the m_i dimensional vector space generated by the eigenspaces of $\square_{Tf}|_{\Omega^i(M)}$ associated to the eigenvalues lying in $[0,1]$, and that one has the finite dimensional subcomplex (5.15) of $(\Omega^*(M), d_{Tf})$,

$$\left(\mathrm{F}^{[0,1]}_{Tf}, d_{Tf}\right) : 0 \longrightarrow \mathrm{F}^{[0,1]}_{Tf,0} \xrightarrow{d_{Tf}} \mathrm{F}^{[0,1]}_{Tf,1} \xrightarrow{d_{Tf}} \cdots \xrightarrow{d_{Tf}} \mathrm{F}^{[0,1]}_{Tf,n} \longrightarrow 0. \tag{6.11}$$

We call $(\mathrm{F}^{[0,1]}_{Tf}, d_{Tf})$ the **Witten instanton complex** associated to Tf.

Now we equip $C^*(W^u)$ with a metric such that for any $x,\ y \in \mathrm{zero}(\nabla f)$ with $x \neq y$, $W^u(x)^*$ and $W^u(y)^*$ are orthogonal to each other, and that

$$\langle W^u(x)^*, W^u(x)^* \rangle = 1$$

for each $x \in \mathrm{zero}(\nabla f)$.

Recall that for any $x \in \mathrm{zero}(\nabla f)$ and $T \geq T_0$, the section $\rho_{x,T} \in \Omega^*(M)$ has been defined in (5.17).

*Compare with [L, Proposition 6].

Definition 6.5 Let J_T be the linear map from $C^*(W^u)$ into $\Omega^*(M)$ such that for any $x \in \mathrm{zero}(\nabla f)$ and $T \geq T_0$,

$$J_T W^u(x)^* = \rho_{x,T}. \tag{6.12}$$

Clearly, J_T is an isometry from $C^*(W^u)$ into $\Omega^*(M)$, which preserves the $\mathbf{Z}$-gradings.

Let $P_T^{[0,1]}$ denote the orthogonal projection from $\Omega^*(M)$ on $\mathrm{F}_{Tf}^{[0,1]}$. Clearly, $P_T^{[0,1]}$ is exactly the orthogonal projection operator $P_T(c)$ defined in Section 5.6 with $c = 1$, as the space $\mathrm{F}_{Tf}^{[0,1]}$ is easily seen to be the same as $E_T(c)$ with $c = 1$ there.

Definition 6.6 Let $e_T : C^*(W^u) \to \mathrm{F}_{Tf}^{[0,1]}$ be given by

$$e_T = P_T^{[0,1]} J_T. \tag{6.13}$$

The following result, which refines Lemma 5.8 significantly, is taken from [BZ1, Theorem 8.8] and [BZ2, Theorem 6.7].

Theorem 6.7 *There exists $c > 0$ such that as $T \to +\infty$, for any $s \in C^*(W^u)$,*

$$\left(e_T - J_T\right) s = O\left(e^{-cT}\right) \|s\|_0 \quad \textit{uniformly on } M. \tag{6.14}$$

In particular, e_T is an isomorphism.

Proof. Let $\delta = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ be the counter-clockwise oriented circle.† Then we can write as in (5.28) that for any $x \in \mathrm{zero}(\nabla f)$ and $T > 0$ large enough,

$$(e_T - J_T) W^u(x)^* = P_T^{[0,1]} \rho_{x,T} - \rho_{x,T}$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_\delta \left((\lambda - D_{Tf})^{-1} - \lambda^{-1} \right) \rho_{x,T} d\lambda$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_\delta (\lambda - D_{Tf})^{-1} \frac{D_{Tf}\rho_{x,T}}{\lambda} d\lambda. \tag{6.15}$$

†Compare with Remark 5.9 on the use of complex coefficients.

For any $p \geq 0$, let $\|\cdot\|_p$ denote the p-th Sobolev norm on $\Omega^*(M)$.

From Proposition 5.4 and the definition of $\rho_{x,T}$, one sees that on a (fixed) sufficiently small open neighborhood of x, one has

$$D_{Tf}\rho_{x,T} = 0. \tag{6.16}$$

By (5.17) and (6.16), for any positive integer p, there is $c_p > 0$ such that as $T \to +\infty$,

$$\|D_{Tf}\rho_{x,T}\|_p = O\left(e^{-c_pT}\right). \tag{6.17}$$

Take $q \geq 1$. Since D is a first order elliptic operator, there exist $C > 0$, $C_1 > 0$ and $C_2 > 0$ such that if $s \in \Omega^*(M)$, then

$$\|s\|_q \leq C_1\left(\|Ds\|_{q-1} + \|s\|_0\right)$$

$$\leq C_1\left(\|(\lambda - D_{Tf})s\|_{q-1} + C_2T\|s\|_{q-1} + \|s\|_0\right)$$

$$\leq CT^q\left(\|(\lambda - D_{Tf})s\|_{q-1} + \|s\|_0\right), \tag{6.18}$$

where the last inequality follows from an induction argument.

On the other hand, by (5.27) one deduces easily that there exists $C' > 0$ such that for $\lambda \in \delta$, $s \in \Omega^*(M)$ and $T > 0$ large enough,

$$\left\|(\lambda - D_{Tf})^{-1}s\right\|_0 \leq C'\|s\|_0. \tag{6.19}$$

By (6.18) and (6.19), there exists $C'' > 0$ such that if $T > 0$ is large enough,

$$\left\|(\lambda - D_{Tf})^{-1}s\right\|_q \leq CT^q\left(\|s\|_{q-1} + C'\|s\|_0\right) \leq C''T^q\|s\|_{q-1}. \tag{6.20}$$

By (6.17) and (6.20), there exists $c_q > 0$ such that when $T > 0$ is large enough,

$$\left\|(\lambda - D_{Tf})^{-1}D_{Tf}\rho_{x,T}\right\|_q = O\left(e^{-c_qT}\right), \text{ uniformly on } \lambda \in \delta. \tag{6.21}$$

Using (6.21) and Sobolev's inequality (cf. [W, Corollary 6.22(b)]), we see that there exists $c > 0$ such that

$$\left|(\lambda - D_{Tf})^{-1}D_{Tf}\rho_{x,T}\right| \leq O\left(e^{-cT}\right), \text{ uniformly on } M. \tag{6.22}$$

By (6.15) and (6.22), (6.14) holds for any $s = W^u(x)^*$ with $x \in \text{zero}(\nabla f)$. It then clearly holds for any $s \in C^*(W^u)$.

Since J_T is an isometry from $C^*(W^u)$ into $\Omega^*(M)$, from (6.14) one sees easily that e_T is an isomorphism when $T > 0$ is large enough. $\square$

6.4 The Map $P_{\infty,T}e_T$

Recall from (6.10) that the de Rham map

$$P_\infty : \alpha \in \Omega^*(M) \longrightarrow \sum_{x\in\mathrm{zero}(\nabla f)} [W^u(x)]^* \int_{\overline{W^u}(x)} \alpha \in C^*(W^u) \qquad (6.23)$$

is a chain homomorphism between the complexes.

From (5.5), (5.15) and (6.23), one verifies easily that if $T > 0$ is large enough, then the map

$$P_{\infty,T} : \mathrm{F}^{[0,1]}_{Tf} \longrightarrow C^*(W^u)$$

defined by

$$P_{\infty,T} : \alpha \mapsto P_\infty e^{Tf}\alpha \qquad (6.24)$$

is also a chain homomorphism of complexes. That is, when acting on $\mathrm{F}^{[0,1]}_{Tf}$, one has

$$P_{\infty,T}d_{Tf} = \widetilde{\partial}P_{\infty,T}. \qquad (6.25)$$

Definition 6.8 Let $\mathcal{F} \in \mathrm{End}(C^*(W^u))$ which, for $x \in \mathrm{zero}(\nabla f)$, acts on $[W^u(x)]^*$ by multiplication by $f(x)$. Let $N \in \mathrm{End}(C^*(W^u))$ which acts on $C^i(W^u)$, $0 \le i \le n$, by multiplication by i.

The following result is taken from [BZ2, Theorem 6.11].

Theorem 6.9 *There exists $c > 0$ such that as $T \to +\infty$,*

$$P_{\infty,T}e_T = e^{T\mathcal{F}}\left(\frac{\pi}{T}\right)^{N/2-n/4}\left(1 + O\left(e^{-cT}\right)\right). \qquad (6.26)$$

In particular, $P_{\infty,T}$ is an isomorphism for $T > 0$ large enough.

Proof. Take $x \in \mathrm{zero}(\nabla f)$, $s = W^u(x)^*$.

By (6.23) and (6.24), we get

$$P_{\infty,T}e_T s = \sum_{\substack{y\in\mathrm{zero}(\nabla f)\\ n_f(y)=n_f(x)}} e^{Tf(y)} W^u(y)^* \int_{\overline{W}^u(y)} e^{T(f-f(y))} e_T s. \tag{6.27}$$

Clearly, for any $y \in \mathrm{zero}(\nabla f)$, one has

$$f - f(y) \leq 0 \quad \text{on} \quad \overline{W}^u(y). \tag{6.28}$$

Since by Proposition 6.2(i) the $\overline{W}^u(y)$'s are compact manifolds with conical singularities, by Theorem 6.7 and (6.28), we see that if $y \in \mathrm{zero}(\nabla f)$ with $n_f(y) = n_f(x)$, then

$$\int_{\overline{W}^u(y)} e^{T(f-f(y))} e_T s = \int_{\overline{W}^u(y)} e^{T(f-f(y))} J_T s + O\left(e^{-cT}\right), \tag{6.29}$$

for some $c > 0$.

Since the support of $J_T s$ is included in U_x, using (5.17), (6.8), (6.12) and (6.29), we find that

$$\int_{\overline{W}^u(x)} e^{T(f-f(x))} e_T s = \left(\frac{\pi}{T}\right)^{n_f(x)/2-n/4} \left(1 + O\left(e^{-cT}\right)\right). \tag{6.30}$$

Take now $y \in \mathrm{zero}(\nabla f)$.

By Proposition 6.2(ii) we know that $\overline{W}^u(y) \setminus W^u(y)$ is a union of certain $\overline{W}^u(y')$, with $n_f(y') < n_f(y)$. Thus we find that for $y \in \mathrm{zero}(\nabla f)$ with $y \neq x$ and $n_f(y) = n_f(x)$, then

$$x \notin \overline{W}^u(y). \tag{6.31}$$

From (5.17), (6.12) and (6.31), we deduce that there is $c' > 0$ such that if $y \in \mathrm{zero}(\nabla f)$ with $y \neq x$ and $n_f(y) = n_f(x)$, then

$$J_T s = O\left(e^{-c'T}\right) \quad \text{on} \quad \overline{W}^u(y). \tag{6.32}$$

Using (6.28) and (6.32), we see that if $y \in \mathrm{zero}(\nabla f)$ with $y \neq x$ and $n_f(y) = n_f(x)$, then

$$\int_{\overline{W}^u(y)} e^{T(f-f(y))} J_T s = O\left(e^{-c'T}\right). \tag{6.33}$$

From (6.27), (6.29), (6.30) and (6.33), one gets (6.26) easily.

The proof of Theorem 6.9 is completed. □

We refer to the paper of Bismut and Goette [BL] for a geneneralization of the results in above sections to the case of fibrations.

6.5 An Analytic Proof of Theorem 6.4

Recall from (6.25) that when $T > 0$ is large enough,

$$P_{\infty,T} : \left(\mathrm{F}_{Tf}^{[0,1]}, d_{Tf}\right) \longrightarrow \left(C^*(W^u), \widetilde{\partial}\right)$$

is a chain homomorphism. Thus it induces a homomorphism on cohomology groups

$$P_{\infty,T}^{H} : H^*\left(\mathrm{F}_{Tf}^{[0,1]}, d_{Tf}\right) \longrightarrow H^*\left(C^*(W^u), \widetilde{\partial}\right).$$

On the other hand, by Theorem 6.9, when $T > 0$ is large enough, $P_{\infty,T}$ is an isomorphism. Thus by (6.25) again one sees that

$$P_{\infty,T}^{-1} : \left(C^*(W^u), \widetilde{\partial}\right) \longrightarrow \left(\mathrm{F}_{Tf}^{[0,1]}, d_{Tf}\right)$$

is also a chain homomorphism, which induces a homomorphism on cohomology groups

$$\left(P_{\infty,T}^{-1}\right)^{H} : H^*\left(C^*(W^u), \widetilde{\partial}\right) \longrightarrow H^*\left(\mathrm{F}_{Tf}^{[0,1]}, d_{Tf}\right).$$

Clearly, $P_{\infty,T}^{H}$ and $(P_{\infty,T}^{-1})^{H}$ are inverse to each other. Thus $P_{\infty,T}$ induces a canonical isomorphism between $H^*(C^*(W^u), \widetilde{\partial})$ and $H^*(\mathrm{F}_{Tf}^{[0,1]}, d_{Tf})$ which clearly preserves the $\mathbf{Z}$-gradings.

Theorem 6.4 then follows easily from Proposition 5.3 and from (6.24). □

6.6 References

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Chapter 7

Atiyah Theorem on Kervaire Semi-characteristic

Recall that in Chapter 4 we have proved the Poincaré-Hopf index formula (4.11) by making use of the deformation introduced by Witten [W]. Now if one changes V to $-V$ in (4.11), one sees easily that the left hand side does not change, while the right hand side will change by a factor $(-1)^{\dim M}$. As a consequence, if $\dim M$ is odd, the Euler characteristic $\chi(M)$ vanishes. On the other hand, a theorem due to Hopf (cf. [S]) states that if a closed manifold M has vanishing Euler characteristic, then there exists a nowhere zero vector field on M.Thus, there always exists a nowhere zero vector field on an odd dimensional closed manifold.

In this chapter we will discuss the following result due to Atiyah [A], which considers the possibility of the existence of two linearly independent vector fields on $4q+1$ dimensional manifolds.*

Atiyah vanishing theorem *If there exist two linearly independent vector fields on a $4q+1$ dimensional oriented closed manifold, then the Kervaire semi-characteristic of this manifold vanishes.*

In this chapter, we will show that this result can also be proved by using the Witten type deformations similar to what has been used in Chapter 4.

We will start with the definition of the Kervaire semi-characteristic.

*Dupont [D] has proved that there always exist three linearly independent vector fields on a $4q+3$ dimensional oriented closed manifold. This generalizes the classical three dimensional result of Stiefel.

7.1 Kervaire Semi-characteristic

Let M be a $4q+1$ dimensional smooth closed oriented manifold. By definition, the **Kervaire semi-characteristic** of M, denoted by $k(M)$, is an element in $\mathbf{Z}_2$ defined by

$$k(M) \equiv \sum_{i=0}^{2q} \dim H^{2i}_{\mathrm{dR}}(M;\mathbf{R}) \mod 2\mathbf{Z}. \tag{7.1}$$

One may think of $k(M)$ as a mod 2 analogue of the Euler characteristic on odd dimensional manifolds. In particular, it admits an analytic interpretation via the Hodge decomposition theorem.

We first describe this analytic interpretation, which is due to Atiyah and Singer [AS], as follows.

Take a metric g^{TM} on TM. Let $e_1, \ldots, e_{4q+1}$ be a (local) oriented orthonormal basis of TM.

We will use the same notation for Clifford actions and so on as in Chapter 4.

Definition 7.1 Let D_{Sig} be the **Signature operator** defined by

$$D_{\mathrm{Sig}} = \widehat{c}(e_1)\cdots\widehat{c}(e_{4q+1})\,(d+d^*) : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{even}}(M). \tag{7.2}$$

Clearly, the operator D_{Sig} is a well-defined first order elliptic differential operator. Moreover, by using (4.13) and (4.16), one verifies directly that D_{Sig} is *skew-adjoint*. That is, for any $s\ , s' \in \Omega^{\mathrm{even}}(M)$,

$$\langle D_{\mathrm{Sig}}s, s'\rangle = -\langle s, D_{\mathrm{Sig}}s'\rangle\,. \tag{7.3}$$

On the other hand, by Corollary 4.4, which is a consequence of the Hodge decomposition theorem, and by (4.8), one has

$$\dim(\ker D_{\mathrm{Sig}}) = \sum_{i=0}^{2q} \dim H^{2i}_{\mathrm{dR}}(M;\mathbf{R}). \tag{7.4}$$

Now for any skew-adjoint elliptic differential operator D, following Atiyah and Singer [AS], one can define an element in $\mathbf{Z}_2$, which is called the **mod 2 index** of D, as follows,

$$\mathrm{ind}_2\, D \equiv \dim(\ker D) \mod 2\mathbf{Z}. \tag{7.5}$$

Furthermore, Atiyah and Singer showed that this mod 2 index is a homotopy invariant. That is, if $D(u)$, $0 \leq u \leq 1$, is a smooth family of skew-adjoint elliptic differential operators on a closed manifold, then†

$$\operatorname{ind}_2 D(1) = \operatorname{ind}_2 D(0) \quad \text{in} \quad \mathbf{Z}_2. \tag{7.6}$$

From (7.1), (7.4) and (7.5), one can write Atiyah-Singer's analytic interpretation of the Kervaire semi-characteristic of M as follows,

$$k(M) = \operatorname{ind}_2 D_{\mathrm{Sig}}. \tag{7.7}$$

7.2 Atiyah's Original Proof

Let V_1, $V_2 \in \Gamma(TM)$ be two smooth vector fields on M. We assume that they are linearly independent over M. That is, for any $x \in M$, $V_1(x)$ and $V_2(x)$ are linearly independent in T_xM. Following Atiyah [A], we now show that under this situation, one has $k(M) = 0$ in $\mathbf{Z}_2$.

Without loss of generality, we take a metric g^{TM} such that for any $x \in M$, $V_1(x)$ and $V_2(x)$ are orthogonal to each other, and that $V_1(x)$ and $V_2(x)$ are of norm one.

Following [A], we construct the following differential operator

$$D' = \frac{1}{2}\left(D_{\mathrm{Sig}} + \widehat{c}(V_1)\,\widehat{c}(V_2)\,D_{\mathrm{Sig}}\widehat{c}(V_2)\,\widehat{c}(V_1)\right). \tag{7.8}$$

By (4.16) and (7.2), one has

$$D_{\mathrm{Sig}} = \widehat{c}(e_1)\cdots\widehat{c}(e_{4q+1}) \sum_{i=1}^{4q+1} c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)}. \tag{7.9}$$

From (4.13), (7.8) and (7.9), one deduces directly that

$$D' = D_{\mathrm{Sig}} + \frac{1}{2}\widehat{c}(e_1)\cdots\widehat{c}(e_{4q+1}) \sum_{i=1}^{4q+1} c(e_i)\widehat{c}(V_1)\,\widehat{c}\left(\nabla_{e_i}^{TM} V_1\right)$$

$$+\frac{1}{2}\widehat{c}(e_1)\cdots\widehat{c}(e_{4q+1}) \sum_{i=1}^{4q+1} c(e_i)\widehat{c}(V_1)\,\widehat{c}(V_2)\,\widehat{c}\left(\nabla_{e_i}^{TM} V_2\right)\widehat{c}(V_1). \tag{7.10}$$

†This follows from the easy fact that if a finite dimensional Euclidean space admits a skew-adjoint automorphism, then it is of even dimension.

From (7.10), one sees that D' is a first order elliptic differential operator. On the other hand, by (4.13) and (7.8), one verifies directly that D' is skew-adjoint. Thus, by using (7.10) again, one sees that for any $u \in [0,1]$,

$$D(u) = (1-u)D_{\mathrm{Sig}} + uD' \tag{7.11}$$

is elliptic and skew-adjoint.

By (7.11) and the homotopy invariance of the mod 2 index, one then gets

$$\mathrm{ind}_2\, D_{\mathrm{Sig}} = \mathrm{ind}_2\, D'. \tag{7.12}$$

Now by our assumption on g^{TM} and by (7.8), one verifies directly that $\widehat{c}(V_1)\widehat{c}(V_2)$, which preserves $\Omega^{\mathrm{even}}(M)$, commutes with D'. Thus, $\widehat{c}(V_1)\widehat{c}(V_2)$ preserves the kernel of D'. On the other hand, one checks that

$$\left(\widehat{c}\,(V_1)\,\widehat{c}\,(V_2)\right)^2 = -1. \tag{7.13}$$

By (7.13), $\widehat{c}(V_1)\widehat{c}(V_2)$ forms a comlpex struction on $\ker D'$, which implies that

$$\dim\left(\ker D'\right) \equiv 0 \mod 2\mathbf{Z}. \tag{7.14}$$

From (7.7), (7.12) and (7.14), one gets the vanishing property of $k(M)$.
□

Remark 7.2 Conversely, Atiyah [A] and Atiyah-Dupont [AD] have shown that for a $4q+1$ dimensional oriented closed manifold M, if both $k(M)$ and the $4q$-th Stiefel-Whitney class of TM vanish,‡ then there exist two linearly independent vector fields on M.

7.3 A proof via Witten Deformation

In this section, we present an alternate proof of the Atiyah vanishing theorem by adapting the deformation idea of Witten.

We first give an alternate analytic interpretation of the Kervaire semi-characteristic $k(M)$.

Let g^{TM} be chosen as in the previous section.

In this section, we denote by $V = V_1$ and $X = V_2$.

‡See [MS] for a definition of the Stiefel-Whitney class of vector bundles.

Definition 7.3 ([Z1]) Let $D_V : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{even}}(M)$ be the operator defined by

$$D_V = \frac{1}{2}\left(\widehat{c}(V)\,(d+d^*) - (d+d^*)\,\widehat{c}(V)\right). \tag{7.15}$$

Clearly, D_V is skew-adjoint.

On the other hand, by using (4.13) and (4.16), one verifies directly that

$$D_V = \widehat{c}(V)\,(d+d^*) - \frac{1}{2}\sum_{i=1}^{4q+1} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} V\right), \tag{7.16}$$

from which one knows that D_V is also an elliptic differential operator of order one.

The following result, which is taken from [Z1], shows that the mod 2 index of D_V gives an alternate analytic interpretation of $k(M)$.

Theorem 7.4 *The following identity in* $\mathbf{Z}_2$ *holds,*

$$\mathrm{ind}_2\, D_V = k(M). \tag{7.17}$$

Proof. Let $D'' : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{even}}(M)$ be the elliptic differential operator defined by

$$D'' = D_{\mathrm{Sig}} - \frac{1}{2}\widehat{c}(e_1)\cdots\widehat{c}(e_{4q+1})\widehat{c}(V)\sum_{i=1}^{4q+1} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} V\right). \tag{7.18}$$

Since $V \in \Gamma(TM)$ is of norm one over M, one sees that for any integer i such that $1 \le i \le 4q+1$,

$$\left\langle V, \nabla_{e_i}^{TM} V\right\rangle = 0.$$

Thus, by (4.13),

$$\widehat{c}(V)\widehat{c}\left(\nabla_{e_i}^{TM} V\right) + \widehat{c}\left(\nabla_{e_i}^{TM} V\right)\widehat{c}(V) = 0. \tag{7.19}$$

From (4.13), (7.3) and (7.19), one verifies easily that D'' is also skew-adjoint. Thus, by the homotopy invariance property of the mod 2 index, one has

$$\mathrm{ind}_2\, D'' = \mathrm{ind}_2\, D_{\mathrm{Sig}}. \tag{7.20}$$

On the other hand, by (4.13), (7.2), (7.16) and (7.18), one verifies directly that

$$\ker D'' = \ker\left(\widehat{c}(e_1)\cdots\widehat{c}(e_{4q+1})\left(d+d^*-\frac{1}{2}\widehat{c}(V)\sum_{i=1}^{4q+1}c(e_i)\widehat{c}\left(\nabla^{TM}_{e_i}V\right)\right)\right)$$

$$= \ker\left(\widehat{c}(V)\left(d+d^*-\frac{1}{2}\widehat{c}(V)\sum_{i=1}^{4q+1}c(e_i)\widehat{c}\left(\nabla^{TM}_{e_i}V\right)\right)\right)$$

$$= \ker D_V. \tag{7.21}$$

From (7.5) and (7.21), one gets,

$$\mathrm{ind}_2\, D'' = \mathrm{ind}_2\, D_V. \tag{7.22}$$

From (7.7), (7.20) and (7.22), one gets (7.15). □

Next, we introduce a deformation of D_V by using the second vector field X.

Definition 7.5 ([Z2]) For any $T \in \mathbf{R}$, let $D_{V,T} : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{even}}(M)$ be the operator defined by

$$D_{V,T} = D_V + T\widehat{c}(V)\widehat{c}(X). \tag{7.23}$$

Remark 7.6 Since V and X are orthogonal to each other, by (4.13), (7.15) and (7.23), one can also write $D_{V,T}$ as

$$D_{V,T} = \frac{1}{2}\left(\widehat{c}(V)\left(d+d^*+T\widehat{c}(X)\right) - \left(d+d^*+T\widehat{c}(X)\right)\widehat{c}(V)\right). \tag{7.24}$$

In view of (4.17), one may regard $D_{V,T}$ as a Witten type deformation of D_V.

Clearly, $D_{V,T}$ is elliptic and skew-adjoint.

By Theorem 7.4 and the homotopy invariance property of the mod 2 index, one gets that for any $T \in \mathbf{R}$,

$$\mathrm{ind}_2\, D_{V,T} = \mathrm{ind}_2\, D_V = k(M). \tag{7.25}$$

We will prove the vanishing of $k(M)$ by studying the behaviour of $\ker D_{V,T}$ as $T \to \infty$.

We first establish a Bochner type formula for $-D_{V,T}^2$.

Proposition 7.7 *The following identity holds,*

$$-D_{V,T}^2 = -D_V^2 + T \sum_{i=1}^{4q+1} \left(c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} X\right) - \left\langle \nabla_{e_i}^{TM} X, V \right\rangle c(e_i)\widehat{c}(V) \right)$$

$$+T^2|X|^2. \tag{7.26}$$

Proof. By (4.13), (7.16) and (7.23), one can rewrite $D_{V,T}$ as

$$D_{V,T} = \widehat{c}(V) \left(d + d^* - \frac{1}{2}\widehat{c}(V) \sum_{i=1}^{4q+1} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} V\right) + T\widehat{c}(X) \right). \tag{7.27}$$

From (4.13), (7.15) and (7.27), one deduces that

$$-D_{V,T}^2 = \left(d + d^* - \frac{1}{2}\widehat{c}(V) \sum_{i=1}^{4q+1} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} V\right) + T\widehat{c}(X) \right)^2$$

$$= -D_V^2 + T\left((d+d^*)\,\widehat{c}(X) + \widehat{c}(X)\,(d+d^*) \right)$$

$$-T \sum_{i=1}^{4q+1} \left\langle \nabla_{e_i}^{TM} X, V \right\rangle c(e_i)\widehat{c}(V) + T^2|X|^2. \tag{7.28}$$

From (7.28) and by proceeding as in the proof of (4.19), one gets (7.26). □

We can now prove the Atiyah vanishing theorem as follows.

Since $|X| = 1$ over M by our assumption, one sees easily that there exists $T_0 > 0$ such that when $T \geq T_0$,

$$T \sum_{i=1}^{4q+1} \left(c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} X\right) - \left\langle \nabla_{e_i}^{TM} X, V \right\rangle c(e_i)\widehat{c}(V) \right) + T^2|X|^2 > 0. \tag{7.29}$$

On the other hand, since D_V is skew-adjoint, $-D_V^2$ is a nonnegative operator. Combining this fact with (7.26) and (7.29), one sees that when

$T \geq T_0$, $-D_{V,T}^2$ is a positive operator, which implies that

$$\ker D_{V,T} = \{0\}. \tag{7.30}$$

From (7.25), (7.30) and the definition of the mod 2 index, one gets the vanishing property of $k(M)$. □

7.4 A Generic Counting Formula for $k(M)$

The proof in the previous section has the advantage that it also leads to a generic counting formula for $k(M)$ in a way similar to what the Poincaré-Hopf formula is for the Euler characteristic.

To state this counting formula, we recall that on the given $4q+1$ dimensional smooth oriented closed manifold M, by the result of Hopf mentioned in the beginning of this chapter, there always exists a nowhere zero vector field V of M.

Let $[V]$ denote the one dimensional vector bundle generated by V.

We consider the quotient bundle $TM/[V]$, which is a $4q$ dimensional vector bundle over M.

Take a transversal section X of $TM/[V]$, which always exists by elementary result in differential topology.

Since the rank of $TM/[V]$ is $4q$, and M is of dimension $4q+1$, one knows that the zero set of X, denoted by zero(X), consists of disjoint one dimensional closed submanifolds (i.e., circles) in M.

Let $TM/[V]$ be equipped with a Euclidean metric.

Take one circle F in zero(X). At any point $y \in F$, the transversal section X induces an automorphism of $T_yM/[V_y]$, which is the restiction of $TM/[V]$ at y.

By the linear algebra result Lemma 4.8, one can determine a one dimensional linear subspace in $\Lambda^*((T_yM/[V_y])^*)$. Moreover, these linear subspaces form a real line bundle, denoted by $o_F(X)$, over F.

Clearly, as a topological line bundle over F, $o_F(X)$ does not depend on the Euclidean metric on $TM/[V]$.

We define a mod 2 index, denoted by $\mathrm{ind}_2(X, F)$, on F by

$$\mathrm{ind}_2\,(X, F) = 1 \quad \text{if } o_F(X) \text{ is orientable over } F \tag{7.31}$$

and

$$\mathrm{ind}_2\,(X,F)=0 \quad \text{if } o_F(X) \text{ is nonorientable over } F. \tag{7.32}$$

We can now state the generic counting formula for $k(M)$, which is taken from [Z2], as follows.

Theorem 7.8 *The following identity in $\mathbf{Z}_2$ holds,*

$$k(M)=\sum_{F\in \mathrm{zero}(X)} \mathrm{ind}_2\,(X,F). \tag{7.33}$$

The basic strategy of the proof of Theorem 7.8 is the same as that of the proof we presented in Chapter 4 for the Poincaré-Hopf formula: one first apply the Bochner type formula (7.26) to localize everything to a sufficiently small open neighborhood of zero(X), and then completing the proof by making use of properties of harmonic oscillators in this small neighborhood. One notable difference is that since here zero(X) consists of circles instead of isolated points, the analysis of harmonic oscillators will lie in the normal spaces to zero(X) in TM, instead of in whole tangent spaces. We refer to the article [Z2] for more details.

Remark 7.9 It is interesting that while the Euler characteristic can be computed by counting isolated zero points of vector fields (cf. (4.11)), here the Kervaire semi-characteristic is computed by counting *circles.*

7.5 Non-multiplicativity of $k(M)$

To conclude this chapter, we apply Theorem 7.8 to give an analytic proof of a non-multiplicativity result of Atiyah and Singer [AS] on the Kervaire semi-characteristic.

We use the same assumptions and notation as in the previous section. We further assume in this section that $H^1(M;\mathbf{Z}_2)$, the first singular cohomology of M with $\mathbf{Z}_2$ coefficient, is nonzero.

Take a *nonzero* element $\alpha\in H^1(M;\mathbf{Z}_2)$. Let

$$\pi_\alpha : \widetilde{M}_\alpha \to M$$

be the double covering determined by α.

Let $w_{4q}(TM) \in H^{4q}(M; \mathbf{Z}_2)$ be the $4q$-th Stiefel-Whitney class of the tangent vector bundle of M.

The non-multiplicativity theorem of Atiyah and Singer can be stated as follows.

Theorem 7.10 *The following identity in* $\mathbf{Z}_2$ *holds,*

$$k\left(\widetilde{M_\alpha}\right) = \langle \alpha \cdot w_{4q}(TM), [M] \rangle . \tag{7.34}$$

Proof. Recall that V is a nowhere zero vector field on M and X is a transversal section of $TM/[V]$. Let $\widetilde{V} = \pi_\alpha^* V$ and $\widetilde{X} = \pi_\alpha^* X$ be the pull-back vector fields of V and X on $\widetilde{M_\alpha}$ respectively. Then $\widetilde{X}$ is a transversal section of $T\widetilde{M_\alpha}/[\widetilde{V}]$.

Clearly, the zero set zero$(\widetilde{X})$ of $\widetilde{X}$ is exactly $\pi_\alpha^{-1}(\text{zero}(X))$.

Let L_α be the real line bundle over M which is determined by α. That is, L_α is the (unique) line bundle over M such that $w_1(L_\alpha) \in H^1(M; \mathbf{Z}_2)$, the first Stiefel-Whitney class of L_α, equals to α.

For any connected component F, which is a circle, in zero(X), there occur two possibilities for $\pi_\alpha^{-1}(F)$:

(i) If $L_\alpha|_F$ is orientable, then $\pi_\alpha^{-1}(F)$ consists of two disjoint circles $\widetilde{F}_1$ and $\widetilde{F}_2$. Moreover, the restrictions of the pull-back line bundle $\pi_\alpha^*(o_F(X))$ on $\widetilde{F}_1$ and $\widetilde{F}_2$ have the same orientability. In summary, in this case one has

$$\mathrm{ind}_2\left(\widetilde{X}, \pi_\alpha^{-1}F\right) = \mathrm{ind}_2\left(\widetilde{X}, \widetilde{F}_1\right) + \mathrm{ind}_2\left(\widetilde{X}, \widetilde{F}_2\right) = 0. \tag{7.35}$$

(ii) If $L_\alpha|_F$ is non-orientable, then $\pi_\alpha^{-1}(F)$ is connected and

$$\pi_\alpha : \pi_\alpha^{-1}(F) \to F$$

is a double covering between circles. In this case, $\pi_\alpha^*(o_F(X))$ is orientable over $\pi_\alpha^{-1}(F)$, and we get

$$\mathrm{ind}_2\left(\widetilde{X}, \pi_\alpha^{-1}F\right) = 1. \tag{7.36}$$

From (7.35), (7.36) and by using Theorem 7.8, one gets immediately that $k(\widetilde{M_\alpha})$ equals to the number of connected components of zero(X) on which the restriction of the line bundle L_α is non-orientable.

Now since by elementary obstruction theory (cf. [MS]), $[\text{zero}(X)] \in$

$H_{4q}(M, \mathbf{Z}_2)$ is dual to $w_{4q}(TM)$, one finds finally that

$$k\left(\widetilde{M}_\alpha\right) = \sum_{F \in \mathrm{zero}(X)} \langle w_1\left(L_\alpha|_F\right), [F]\rangle = \langle \alpha \cdot w_{4q}(TM), [M]\rangle\,,$$

which is exactly (7.34).

The proof of Theorem 7.10 is completed. □

As was pointed out by Atiyah and Singer in [AS], Theorem 7.8 shows that the Kervaire semi-characteristic is a subtle invariant which, to be different with respect to the Euler characteristic, does not admit a direct differential geometric interpretation.

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